Computer Graphics and HCI Group
AG Computergrafik und HCI

FACHBEREICH INFORMATIK

Geometric Modelling Summer 2017

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Notes:
Due: none
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If you have questions or encounter any problems, feel free to visit me in room 36-233 or to send an email to karer@rhrk.uni-kl.de.

Sheet No.1: Analytic Geomrty, Projective Geometry, and Affine Geometry

1) Higher Order Vector Spaces

a) Let V be a vector space. Prove that the wedge product $\vec{a} \wedge \vec{b}$ of $\vec{a} \in \Lambda^k(V)$ and $\vec{b} \in \Lambda^l(V)$ is in $\Lambda^{k+l}(V)$.

b) Using the notation of a), prove that $\vec{a} \wedge \vec{b} = (-1)^{kl}\vec{b} \wedge \vec{a}$.

c) Prove that the exterior product is associative.

2) Objects in P^2

Your task is to project objects from a given plane F through an eye point O into a canvas plane C that is a subset of P^2. Let the plane F be denoted by $z = 0$, the canvas C by $x = 0$ and the eye $O = (1, 1, 1)^T$ (as in the exercise class).

a) **Parabola**: Consider the parabola $y = x^2$ in F. Determine its image in C by perspective projection through O. Draw a sketch of the resulting image.

b) **Circle**: What is the image in C of the circle in F with radius 2 and center $(1, 1) \in F$? Draw a sketch of the resulting image.

3) Principle of Duality

Prove: Every line in a projective space is the intersection of at least three hyperplanes.
To do so, apply the principle of duality (i.e. prove the dual statement).

4) Affine Transformations

In class, we derived the rotation around the z-axis with respect to an arbitrary point. Derive the map for the rotation with respect to an arbitrary point in 3-dimensional space but around an arbitrary axis.

5) Elliptic and Hyperbolic Spaces

In the lecture, it is stated that three lines in the hyperbolic plane uniquely define a triangle. The slides show a picture of a triangle in the Poincare disk model. What would the situation look like in the Beltrami-Klein model? Does this also hold for elliptic spaces? Provide reasons for your answers.

Sheet No.2: Affine Geometry and Curve Theory

1) Some Proofs:

Prove the following statements:

a) A planar curve is a line iff its curvature vanishes.

b) The center point of the osculating circle is stationary exactly where the curvature is stationary.

2) Arc Length Parameterization

Consider the following helix:

$$t\colon [0, 2\pi] \to \mathbb{R}^3, \quad t \longmapsto (\alpha\cos(t), \alpha\sin(t), \beta t)^T$$

Compute the arc length parameterization and prove that the normal unit vector in every curve point crosses the axis of the helix' cylinder.

3) Coordinate Lines

Consider the following helicoid:

$$\begin{aligned} X &: \mathbb{R}\setminus\{0\}\times\mathbb{R} \longrightarrow \mathbb{R}^3 \\ (u, v) &\longmapsto (u\cos v, u\sin v, C\cdot v) \qquad ; C \in \mathbb{R} \end{aligned}$$

Determine the surface's coordinate lines. What special kind of curves are they? Also determine the surface's unit normal.

4) Modelling Surface Strips

Provide parameterizations for the following surface strips and plot or draw your result:

a) A flat strip winding up a cylinder wall in a spiral shape.

b) A Möbius Strip. Make sure, you guarantee at least C^1-continuity!

5) Planarity of Space Curves

Prove: A space curve is planar iff its torsion vanishes.

6) Fundamental Theorem of Curve Theory

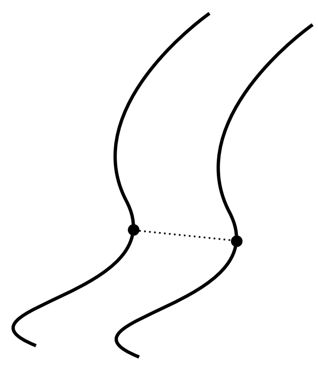
In the lecture, we stated that a curvature and torsion would uniquely determine a curve up to Euclidian motions. Convince yourself of this fact by coming up with an example for each of the following types of curve. Then try to draw your example in differently rotated and scaled 3-dimensional Cartesian coordinate systems.

a) A curve with non-vanishing curvature and vanishing torsion.

b) A curve with non-vanishing curvature and torsion.

7) Parallel Curves

Let $X, \tilde{X}$ be two regular parameterized C^3 curves with non-vanishing curvature. Moreover, for every $t \in [a, b]$, let the line connecting $X(t)$ and $\tilde{X}(t)$ be parallel to the principal normal vectors of both curves.
Prove that the distance between $X(t)$ and $\tilde{X}(t)$ and the angle between the tangent vectors of both curves are constant in t.



Sheet No.3: Curve Theory

1) Coordinate Lines

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$$(u, v) \longmapsto (u \cos v, u \sin v, C \cdot v) \qquad ; C \in \mathbb{R}$$

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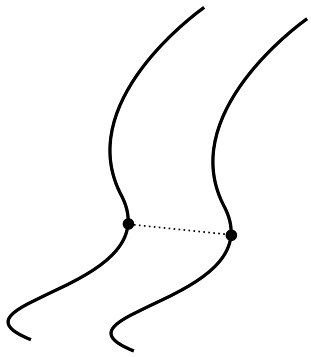

Sheet No.3: Surface Theory

1) Principal Curvature

Let $X : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by:

$$X(u, v) = (u \cos v, u \sin v, hv)^T \qquad ; (u, v) \in \mathbb{R}^2$$

where $h = const. \neq 0$.
Calculate the principal curvatures and their directions.

2) Shape Operator, Mean and Gaussian Curvature

Consider the following parameterization of a torus:

$$(\varphi, \psi) \longrightarrow ((P + \rho \cos \psi) \cos \varphi, (P + \rho \cos \psi) \sin \varphi, \rho \sin \psi) \qquad , P > \rho > 0$$

Determine the shape operator (aka. Weingarten map) L, the Gaussian curvature K and the mean curvature H.

3) Fundamental Forms

Let M be a regular surface with p_u and q_u points in the tangent space U_M of M. Then, the third fundamental form is defined as

$$\mathbf{III}(p_u, q_u) = L_U(p_u) \cdot L_U(q_u)$$

where L_U is the shape operator. In terms of the first and second fundamental form, the third fundamental form is given implicitly by $0 = \mathbf{III} - 2H\mathbf{II} + K\mathbf{I}$, where H denotes the mean curvature and K the Gaussian curvature.

Let $y\colon I \to \mathbb{R}^3$ a regular path with $y(t) = (f(t), 0, g(t))^T$ and $X\colon I \times \mathbb{R} \to \mathbb{R}^3$ the rotation surface defined by:

$$X(t, v) = (f(t)\cos v, f(t)\sin v, g(t))^T$$

where $f(t) > 0$.
Calculate the fundamental forms I, II, and III.

4) Offset Curves

Let g be the following piecewise defined closed space curve:

$$\begin{aligned}
g :& \mathbb{R} \to \mathbb{R}^3 \\
X =& \quad (sin(t-1), cos(t-1), -0.5)^T - (1,0,0)^T, \text{ for} t \in [0,1] \\
X =& \quad (sin(t-2), cos(t-2), \tfrac{t-2}{4})^T, \text{ for} t \in [2,3] \\
X =& \quad (-.5, -2, .75)^T - (t-4)(-0.3, 0.2, -0.25)^T, \text{ for} t \in [4,5]
\end{aligned}$$

, where the intervals $[1,2]$, $[3,4]$, and '$[5,0]$' connect the curve segments C^1-continuously by interpolating quadratic Bezier segments.

Pick at least two of these intervals and compute the offset curve obtained from applying an offset of 1 along the following directions:

- in interval $[0,1]$: $(0,0,1)^T$
- in interval $[2,3]$: 1 along the principal normal direction
- in interval $[4,5]$: $(0,1,0)$
- for the rest, interpolate the offset directions in the respective start and end points beginning with the direction in the start point and rotating the vector with constant velocity (i.e. adjust the rotation angle depending on the the arc length)

Challenge: Also do this for the other segments to obtain the complete offset curve. Do *not* use computers or other technical aids. Document every step of your solution and how exactly you apply it.