Geometric Modelling Summer 2017

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Foundations from Projective Geometry

Motivating Considerations

(This follows K.P. Grotemeyer: *Analytic Geometry*.)

Starting with ordinary Cartesian (Euclidean) coordinates $\overline{x}_1$, $\overline{x}_2$, $\overline{x}_3$ of a point P in $\mathbb{R}^3$ ($\mathbb{E}^3$), we introduce four variables x_0, x_1, x_2, x_3 by carrying out the following operations:

$$\overline{x}_1 = \frac{x_1}{x_0}; \quad \overline{x}_2 = \frac{x_2}{x_0}; \quad \overline{x}_3 = \frac{x_3}{x_0}$$

These variables are called **homogeneous coordinates**.

Motivating Considerations

The (Euclidean) point P is mapped to the column vector $\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$

and vice versa every column vector of this kind with $x_0 \neq 0$ is a point in Euclidean space, but:

$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $\begin{pmatrix} cx_0 \\ cx_1 \\ cx_2 \\ cx_3 \end{pmatrix}$ map to the same point if $c \neq 0$.

Motivating Considerations

Points with $x = 0$ have no Euclidean interpretation. They are **points at infinity**. $(0, a_1, a_2, a_3)^T$ can be interpreted as the infinitely far point along the line in the direction from the coordinate offspring to the Euclidean point $(a_1, a_2, a_3)^T$.

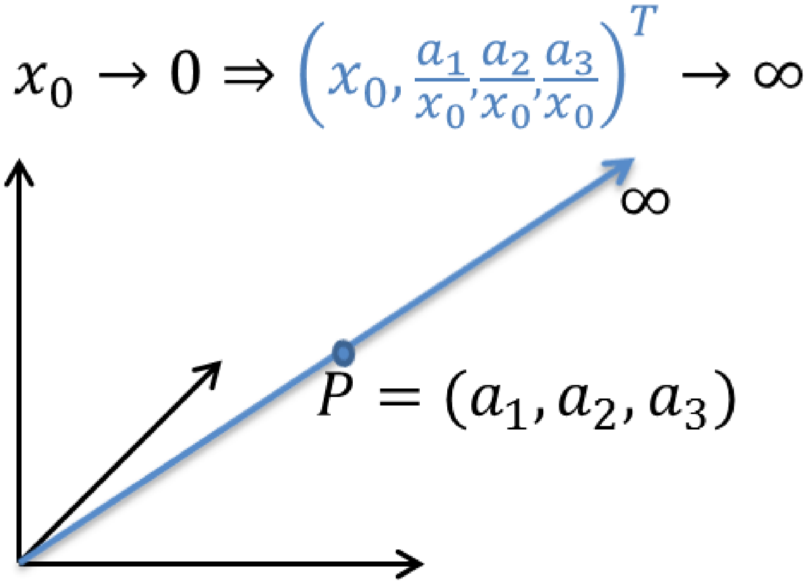


Figure: When x_0 approaches 0, the length of the direction vector approaches infinity.

Motivating Considerations

The equation $x_0 = 0$ describes the **plane at infinity**, the plane that contains all points at infinity.

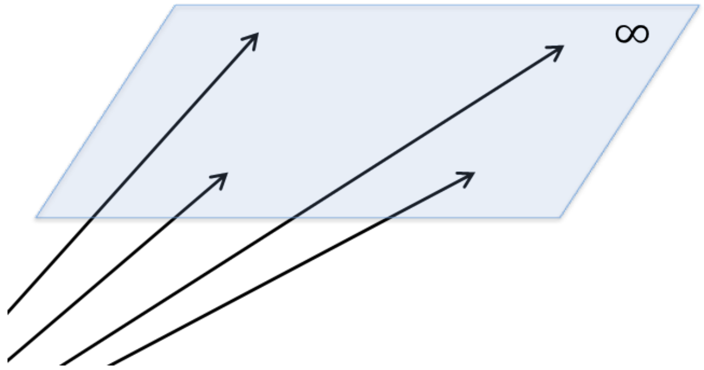


Figure: Plane at infinity: $x_0 = 0$.

$\mathbb{E}^3 +$ plane at infinity $\rightarrow P^3$ **projective space**.
In P^2, all lines intersect in one point, which is a point at infinity in case of Euclidean parallelism, or they are identical. In P^3, all planes intersect in exactly one line – probably a line at infinity – or they are identical.

Motivating Considerations

Line equation in homogeneous coordinates:

$$u_0 x_0 + u_1 x_1 + u_2 x_2 = 0$$

Plane equation in homogeneous coordinates:

$$u_0 x_0 + u_1 x_1 + u_2 x_2 + u_3 x_3 = 0$$

Motivating Considerations

Principle of Duality: Every 4-column has two geometric interpretations: As a point or as a plane.

Point	Plane
Line $\rightarrow$ connection of two points	Line $\rightarrow$ intersection of two planes
connect	intersect
line of points	sheaf of planes

Projective Transformations

The principle of duality can only be applied to results where relations between positions are concerned. In projective geometry, there are no notions of metric properties like length, angle, area, or volume!

Definition: Projective Transformations of P^3, Collineation, Correlation

Projective Transformation: $F : P^3 \to P^3; \quad \tilde{\zeta} = A\zeta; \quad \det A \neq 0.$
Collineation: $\tilde{\zeta}$ and ζ either both represent points or both represent planes.
Correlation: each of $\tilde{\zeta}$ and ζ may represent a point or a plane.

Projective Transformations

Definition: Linear Maps and Frames of Reference in Projective Spaces, Projective Coordinates

Let $\{\vec{\alpha}_i\}$ be linearly independent 4-columns in P^3, i.e. points or planes in P^3.

Figures of 1st order of P^3: $\zeta = \mu_0\vec{\alpha}_0 + \mu_1\vec{\alpha}_1$

Examples: lines of points, sheaf of planes, line as support of a line of points or a sheaf of planes.

Figures of 2nd order of P^3: $\zeta = \mu_0\vec{\alpha}_0 + \mu_1\vec{\alpha}_1 + \mu_2\vec{\alpha}_2$

Examples: plane as support of points (so-called bundle of planes), points as support of a bundle of planes.

Figures of 3rd order of P^3: $\zeta = \mu_0\vec{\alpha}_0 + \mu_1\vec{\alpha}_1 + \mu_2\vec{\alpha}_2 + \mu_3\vec{\alpha}_3$

Examples: all points of P^3, all planes of P^3.

μ_0, μ_1, μ_2, and μ_3 are called **projective coordinates**.

Projective Transformations

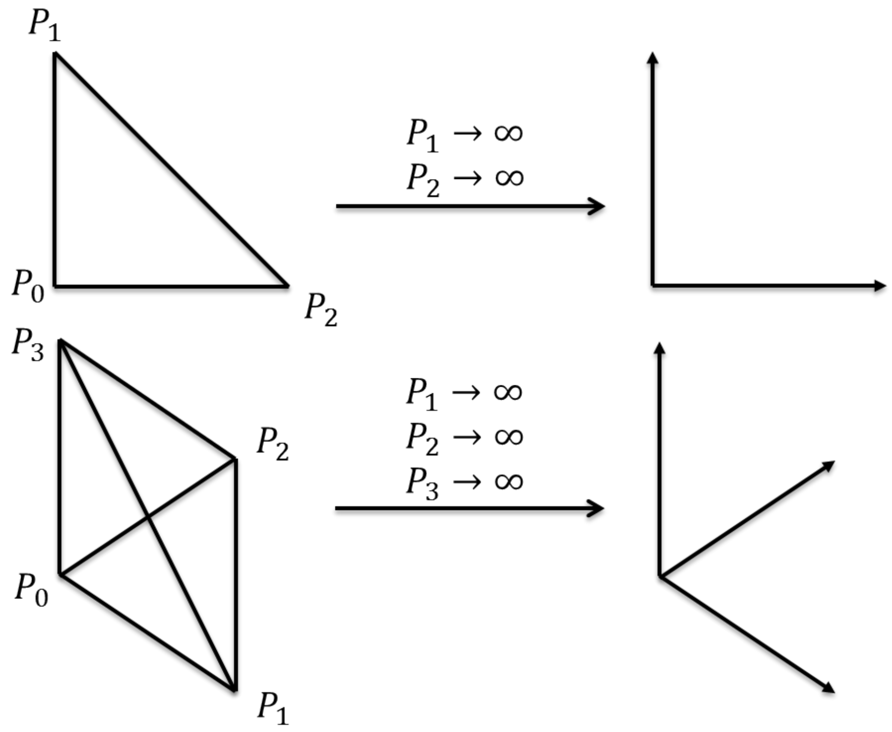
Fundamental elements of these figures:

- 1st order: $\begin{pmatrix}1\\0\end{pmatrix}$; $\begin{pmatrix}0\\1\end{pmatrix}$
- 2nd order: $\begin{pmatrix}1\\0\\0\end{pmatrix}$; $\begin{pmatrix}0\\1\\0\end{pmatrix}$; $\begin{pmatrix}0\\0\\1\end{pmatrix}$
- 3rd order: $\vec{e}_0 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$; $\vec{e}_1 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$; $\vec{e}_2 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$; $\vec{e}_3 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

All the examples mentioned in the definition above (and more figures) can be constructed by linear projective transformations A from these fundamental elements.

Projective Transformations

In P^3, the fundamental elements form a coordinate tetrahedron.
In P^2, a coordinate triangle results.
The Cartesian (Euclidean) coordinate system features the "zero point" $\vec{e}_0$ and $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ as points at infinity.



Projective Transformations

Projective Invariants:

- projective transformations map lines to lines
- a linear figure of order r is is mapped to a figure of same order by a projective transformation. The projective coordinates stay the same
- Let ζ_1, ζ_2, ζ_3, and ζ_4 elements of a 1st-order figure and the columns $\begin{pmatrix}\alpha_0\\ \alpha_1\end{pmatrix}$, $\begin{pmatrix}\beta_0\\ \beta_1\end{pmatrix}$, $\begin{pmatrix}\gamma_0\\ \gamma_1\end{pmatrix}$, and $\begin{pmatrix}\delta_0\\ \delta_1\end{pmatrix}$ the projective coordinates of these figures. The cross ratio $CR(\zeta_0, \zeta_1, \zeta_2, \zeta_3)$ is invariant under projective transformation.

Projective Transformations

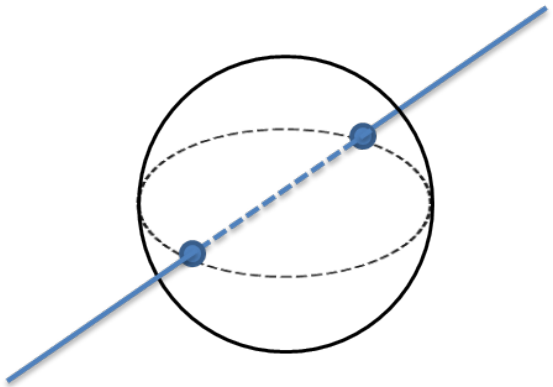
Definition: Cross Ratio

Let ζ_1, ζ_2, ζ_3, and ζ_4 elements of a 1st-order figure and the columns $\begin{pmatrix}\alpha_0\\ \alpha_1\end{pmatrix}$, $\begin{pmatrix}\beta_0\\ \beta_1\end{pmatrix}$, $\begin{pmatrix}\gamma_0\\ \gamma_1\end{pmatrix}$, and $\begin{pmatrix}\delta_0\\ \delta_1\end{pmatrix}$ the projective coordinates of these figures. Their **cross ratio** (also called double ratio) is defined as:

$$CR(\zeta_0, \zeta_1, \zeta_2, \zeta_3) = \frac{\begin{vmatrix}\alpha_0 & \gamma_0\\ \alpha_1 & \gamma_1\end{vmatrix}}{\begin{vmatrix}\beta_0 & \gamma_0\\ \beta_1 & \gamma_1\end{vmatrix}} : \frac{\begin{vmatrix}\alpha_0 & \delta_0\\ \alpha_1 & \delta_1\end{vmatrix}}{\begin{vmatrix}\beta_0 & \delta_0\\ \beta_1 & \delta_1\end{vmatrix}}$$

Projective Transformations

Two (isomorphic) models of P^2:



Points: antipodal point pairs ↔ lines through origin
Lines: great circles ↔ planes through origin

Analytical Structure of Projective Geometry

Definition: Projective Spaces

- Let V be a $(n+1)$-dim. vector space over a field F. The set of one-dim. vector spaces $\Pi K := \{t \cdot \Pi,\ t \in K,\ \Pi \in V, \Pi \neq \vec{0}\}$ is called the **projective space** of **projective dimension** n belonging to V. Notation: $P(V)$ with $\dim P(V) = n$.
- If U is a $(k+1)$-dim. subspace of V, $P(U)$ is called the k-dim **projective subspace** of $P(V)$.
  $k = 0$: "points"; $k = 1$: "lines"; $k = 2$: "planes", ...,
  $k = n - 1$: "hyperplanes"
- A point $P(\vec{r})$ is **incident** with a projective subspace $P(U)$ iff. the vector subspace $\vec{r}$ is contained in U.
- Points $P(\vec{a}_0), \dots, P(\vec{a}_n)$ are called **linearly independent** iff. $\vec{a}_0, \dots, \vec{a}_n$ are linearly independent vectors.

Analytical Structure of Projective Geometry

Comment:
The projective space $P(V)$ is spanned by $n+1$ linearly independent vectors but while in V after picking a base every vector can be associated with a unique $(n+1)$-tuple $(\lambda_0, \ldots, \lambda_n)$, this is not possible in $P(V)$!
An additional norming point is needed to uniquely determine the base vectors up to a common factor $p \in K \setminus \{0\}$.

Analytical Structure of Projective Geometry

Definition: Projective Coordinate Systems

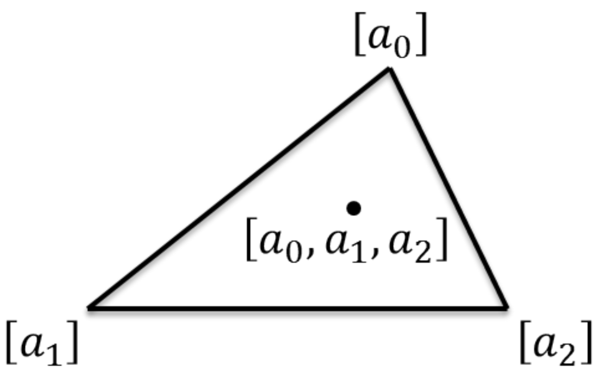
Every $(n+2)$-tuple of points of the projective space $P(V)$ having the property that $n+1$ points are always linearly independent, is called a **projective coordinate system** of $P(V)$.

Theorem

Every projective coordinate system of $P(V)$ can be represented in the form $\{P(\vec{a}_0), P(\vec{a}_1), \ldots, P(\vec{a}_n), P(\vec{a}_0 + \vec{a}_1 + \ldots + \vec{a}_n)\}$, where $\vec{a}_0, \ldots \vec{a}_n$ are a basis of V.

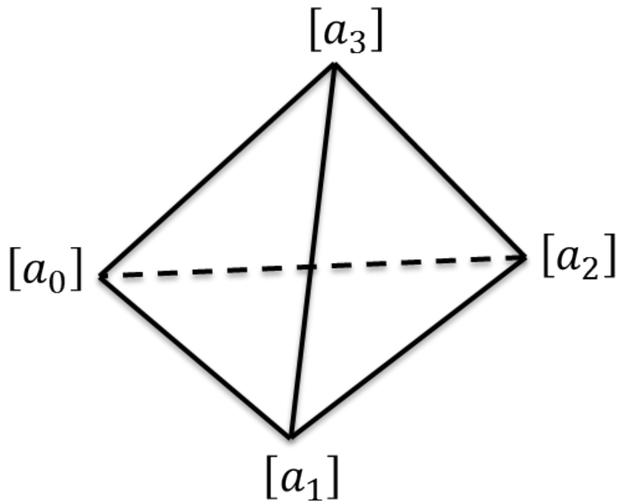
Analytical Structure of Projective Geometry

Projective Plane



coordinate triple

Projective space



coordinate quadruplet

Analytical Structure of Projective Geometry

Comments:
A k-dim. projective subspace $P(U)$ can thus be characterized by $k+1$ linearly independent points $P(\vec{a}_0), \ldots, P(\vec{a}_k)$, but also by $n-k$ linear functions $L_1, \ldots L_{n-k}$ over $V: P(U) = \{P(\vec{r}) \in P : L_1(\vec{r}) = \ldots = L_{n-k}(\vec{r}) = 0\}$.
To every k-dim. subspace of P(V) belongs a $(n-k-1)$-dim. subspace of $P(L(V,K))$. In particular, every hyperplane in $P(V)$ corresponds to a point in $P(L(V,K))$ and every point in $P(V)$ to a hyperplane in $P(L(V,K))$.

Analytical Structure of Projective Geometry

Definition: Dual Vector Spaces

Let V and V^* be vector spaces over a common scalar field F with $\dim V = \dim V^* = n + 1$.
V and V^* are called **dual vector spaces** iff. there is a bilinear map $\langle \cdot, \cdot \rangle : V^* \times V \to K$ which does not degenerate, i.e.

$$\langle \Pi, x \rangle = 0 \; \forall \Pi \in V^* \quad \Rightarrow x = 0$$
$$\langle \Pi, x \rangle = 0 \; \forall x \in V \quad \Rightarrow \Pi = 0$$

Analytical Structure of Projective Geometry

Examples:

1. Let V the Euclidean vector space, $\langle\cdot,\cdot\rangle$ the scalar product.
   $\rightarrow (V, V)$ is a pair of dual vector spaces.
2. Let V the Euclidean vector space,
   $V^* := L(V, K);\ \langle f, x\rangle : f(x)$.
   $\rightarrow (V, L(V, K))$ is a pair of dual vector spaces.
3. Let $V = K^{n+1}, V^* = K^{n+1}; \Pi = (\Pi_0, \ldots \Pi_n)$,
   $x = (x_0, \ldots, x_n), \langle \Pi, x\rangle = sum^n_{i,j=0} c_{ij}\Pi_i x_i$.
   $\rightarrow$ If $det(c_{ij}) \neq 0$, (V, V^*) is a pair of dual vector spaces.

Analytical Structure of Projective Geometry

Definition: Projective Coordinate Systems

Let (V, V^*) be a pair of dual vector spaces. Then, $P(V^*)$ is called **dual** to $P(V)$.

Theorem

A point $P(\vec{r})$ is incident with $P(U)$ and $P(V)$ iff. $\langle \Pi, \vec{r} \rangle = 0$ for all $\Pi \in U^+ = \{\Pi \in V^* : \langle \Pi, \vec{a} \rangle = 0 \; \forall \vec{a} \in V\}$.

Theorem: Duality Principle

If one takes a theorem holding in $P(V)$ and replaces every occurence of "connection" by "intersection", "intersection" by "connection", and "k-dim. subspace" by "$(n-k-1)$-dim subspace, one obtains a new "dual theorem" that holds in $P(V)$.

Analytical Structure of Projective Geometry

Definition: Automorphism

Let V a vector space with $\dim V = n + 1$. Every invertible linear map $A : V \to V$ is called automorphism of V. The set of all automorphisms of V constitute a group called $GL(V)$ resp. $GL(n+1, K)$.
After picking a base for V, every automorphism A of V bijectively associated with a quadratic matrix with non-zero determinant.

Analytical Structure of Projective Geometry

Definition: Group of Projective Transformations

If $A \in GL(n+1, K)$, the map $\alpha : P(V) \to P(V)$ induced by A on $P(V)$ is called a **projective transformation**. The automorphisms A and A' of V induce the same map α on $P(V)$ if $A = c \cdot A'$ for some $c \neq 0$.
The set of all homothetic transformations $H_{n+1} := \{c \cdot Id | c \neq 0\}$ is a normal subgroup (dt. "Normalteiler") of $GL(n+1, K)$. The quotient group (dt. "Faktorgruppe")

$$\mathbb{P} := P.GL(n+1, K) := GL(n+1, K)/H_{n+1}$$

is the **group of the projective transformations** of P(V).

The next chapter will discuss further detail on the group theoretical aspects within the scope of Felix Klein's Erlanger Programm.

Analytical Structure of Projective Geometry

Properties of Projective Transformations:

- bijective
- preserve the cross ratio
- map projective subspaces to projective subspaces of equal dimension

Analytical Structure of Projective Geometry

Properties of Projective Transformations:

- bijective
- preserve the cross ratio
- map projective subspaces to projective subspaces of equal dimension

Analytical Structure of Projective Geometry

Comments:

- Every line-preserving bijective map that preserves the cross ratio from $P(V)$ to itself is a projective transformation.
- The subject of projective geometry is to investigate properties of configurations and figures that are invariant under projective transformations.
- Every line-preserving bijective map of a real projective space whose dimension is at least two is a projective transformation.

Affine Spaces, Elliptic and Hyperbolic Geometry

Affine Spaces

Euclidean Geometry can be formulated by the invariant theories.
Question: What is invariant under Euclidean transformations (rotation, translation)?
Answer: length of vectors, angles between vectors, volumes spanned by vectors
Question: What is invariant under projective transformations?
Answer: dimension of subspaces (linea are mapped to lines etc.), cross ratio
In this chapter, we will discuss affine spaces and transformations as well as elliptic and hyperbolic geometry.

Affine Spaces

Definition: Affine Space

A non-empty set A of elements ("points") is called an **Affine Space**, if there exist a vector space V and a map $\mapsto$ s.t. every pair (p, q) of points with $p, q \in A$ are mapped to exactly one vector $\vec{v} \in V$ s.t.

a) For every $\vec{v} \in V$ and $p \in A$ there is exactly one point $q \in A$ with $(p, q) \mapsto \vec{v}$.

b) If $(p, q) \mapsto \vec{v}$ and $(q, r) \mapsto \vec{w}$, then $(p, r) \mapsto \vec{v} + \vec{w}$.

V is called the vector space belonging to A.
A **base of A** is given by $A : \{0, \vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$, where $0 \in A$ and $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ form a base of V.

Affine Spaces

Definition: Affine Transformation

A map $F : a \rightarrow B$ where A and B are affine spaces is called an **affine transformation**, if it preserves the colinearity of points and the ratios of vectors along a line.
A **coordinate representation** of such a transformation is given by "a linear transformation $+\vec{a}$":

$$\vec{y} = A\vec{x} + \vec{a} \quad \text{where } det(A) \neq 0$$

Affine Spaces

properties of affine transformations:

- preserve collinearity and coplanarity of points (i.e.: lines and planes are preserved)
- preserve ratios of vectors along lines
- preserve parallelity

Affine Spaces

Definition: Affine Group

Let A_n denote an n-dim. affine space. Then, $\mathcal{A}_n := \{F | F : A_n \mapsto A_n, F$ nondegeneerate (i.e. invertible) and affine$\}$ is a group under composition. $\mathcal{A}_n$ is called the **affine group**.

Theorem

The set of all n-row quadratic matrices whose determinant does not vanish constitutes a group under matrix multiplication. If R is the scalar field, this group is denoted $GL(n, R)$. The real affine group is isomorphic to $GL(n, \mathbb{R})$.

Erlangen Program

Felix Klein's Erlangen Program
Felix Klein proposed to employ groups for a unified and clear treatment of geometry. In this view, the geometry belonging to a group of transformations is the theory of invariants of this group. Therefore, the task of affine geometry is to investigate properties that are invariant under affine transformations.

From group theory, "hierarchies of geometry" arise.

Erlangen Program

The **Euclidean motions** are special cases of affine transformations that are not only line-, ratio- and paralell-preserving but also preserve metric quantities (angles, lengths, volumes,...). The **Euclidean group** is a subgroup of the affine group.

Coordinate Representation of the Euclidean Motions

$\vec{y} = B\vec{x} + \vec{a}$ where B is an orthogonal matrix (i.e.) $B^T = B^{-1}$

$det(B) > 0 \rightarrow$ orientation preserving, **direct motions**, rigid motions (examples: rotations, translations)
$det(B) < 0 \rightarrow$ not preserving the orientation, **indirect motions** (example: axis reflections, point reflections in more than 2d)

Erlangen Program

Integration of the Classical Geometries in the Sense of the Erlangen Program:
Consider the subgroup of the projective group (i.e. the corresponding geometry) which fixes a hyper plane. We so to say "tag" this **hyperplane as absolute figure at infinity** and restrict the effect of the subgroup of the transformation group to the points that are not incident with this hyperplane.
This precedure yields affine transformations by restricting the projective transformations to the points that are not at infinity.
The **affine group** then reveals itself as a **subgroup of the projective group**.

Erlangen Program

"The Procedure in Coordinates": P^3 ($x_0 = 0$ is mapped to $y = 0$)

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} a_{00} & 0 & 0 & 0 \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\rightarrow \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} a_{10} \cdot x_0 & a_{11} \cdot x_1 & a_{12} \cdot x_2 & a_{13} \cdot x_3 \\ a_{20} \cdot x_0 & a_{21} \cdot x_1 & a_{22} \cdot x_2 & a_{23} \cdot x_3 \\ a_{30} \cdot x_0 & a_{31} \cdot x_1 & a_{32} \cdot x_2 & a_{33} \cdot x_3 \end{pmatrix}$$

$\rightarrow$ restrict to the points not at infinity (divide by x_0)

$$\rightarrow \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} \frac{x_1}{x_0} & \frac{x_2}{x_0} & \frac{x_3}{x_0} \end{pmatrix}^T + \begin{pmatrix} a_{10} \\ a_{20} \\ a_{30} \end{pmatrix}$$

affine transformation

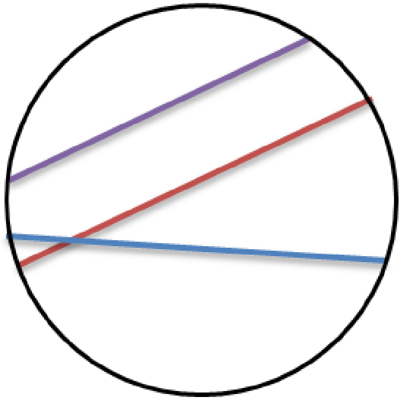
Erlangen Program

Not only the affine (and therefore the Euclidean) geometries can be "integrated" into projective geometry. This proposition does also hold for the hyperbolic and elliptic geometries. Unfortunately we cannot discuss the axiomatic structure of geometry thoroughly in this course.
The common axiomatic foundation of the Euclidean, the elliptic, and the hyperbolic geometries is the so-called *absolute geometry*.
To this fundamental axiomatic system, one of the interchangeable parallel axioms is added:

- elliptical parallel axiom: There is no parallel line.
- Euclidean parallel axiom: There is exactly one parallel line.
- hyperbolic parallel axiom: There are at least two parallel lines.

Elliptic and Hyperbolic Geometry

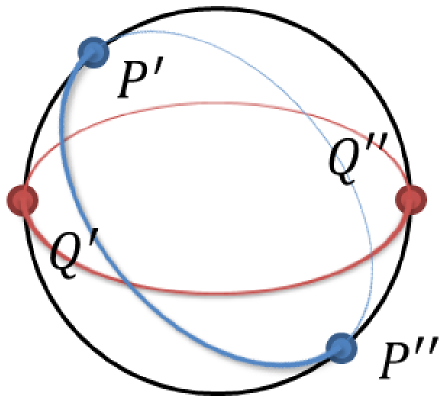
Model of the Hyperbolic Plane (Beltrami-Klein model):



Points: $\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < 1\}$

Lines: $\{\text{secants}\}$

Model of the Elliptic Plane:



Points: $\{(P', P'') | P\ , P'' \in S^2 \wedge x_i' = -x_i''\}$

Lines: $\{\text{great circles}\}$

Elliptic and Hyperbolic Geometry

Important Note: Elliptic and Hyperbolic spaces are metric spaces, i.e. lengths, angles, volumes,... are computable in the sense of elliptical or hyperbolical metrics!

Elliptic and Hyperbolic Geometry

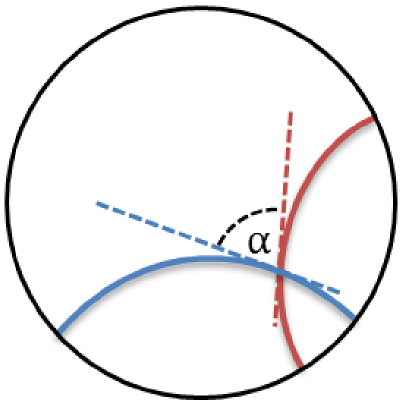
In later chapters, we will discuss the theory of geodesics. This will enable us to measure the lengths of lines on hyperplanes of arbitrary curvature. As elliptic space have a constant positive and hyperbolic spaces have a constant negative curvature, this will also enable us to study lengths in these geometries. Thus, we will not provide proper equations here.
We will discuss the measurement of angles in hyperbolic spaces, though.

Measuring angles in the Beltrami-Klein model is rather difficult, so we first introduce another model of hyperbolic geometry, the Poincaré disk model. After that, we introduce the hyperbolic functions and discuss triangles in the hyperbolic plane.

Elliptic and Hyperbolic Geometry

Poincaré disk model

The Beltrami-Klein model is only conformal in the offspring making it hard to measure angles as their values depend on the position. In the Poincaré disk model, angles are measured like in Euclidean space. The angles between two lines in the Poincaré disk model are the angles between the tangents at the intersection of the two lines.



Points: $\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < 1\}$

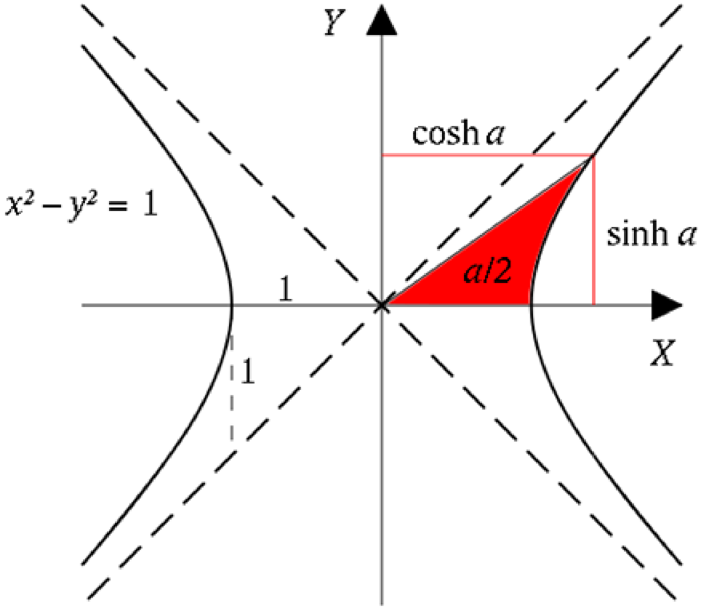
Lines: $\{\text{arcs}\}$

Figure: Two lines (red) and (blue), and the angle between them in the Poincaré disk model.

Elliptic and Hyperbolic Geometry

Hyperbolic Functions

The hyperbolic sine and hyperbolic cosine describe the right half of the unit hyperbola $x^2 - y^2 = 1$ just like sine and cosine describe the unit circle. Moreover, the hyperbolic cosine describes the form of a rope hung up at its two end points.



Elliptic and Hyperbolic Geometry

The hyperbolic functions are given by the following equations:

Definition: Hyperbolic Sine, Hyperbolic Cosine, Hyperbolic Tangent

The hyperbolic sine and hyperbolic cosine are the odd and even components of the exponential function:

- **Hyperbolic Sine**: $\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right) = -i\sin(ix)$
- **Hyperbolic Cosine**: $\sinh x = \frac{1}{2}\left(e^x + e^{-x}\right) = \cos(ix)$
- **Hyperbolic Tangent**: $\tanh x = \frac{\sinh x}{\cosh x}$

Elliptic and Hyperbolic Geometry

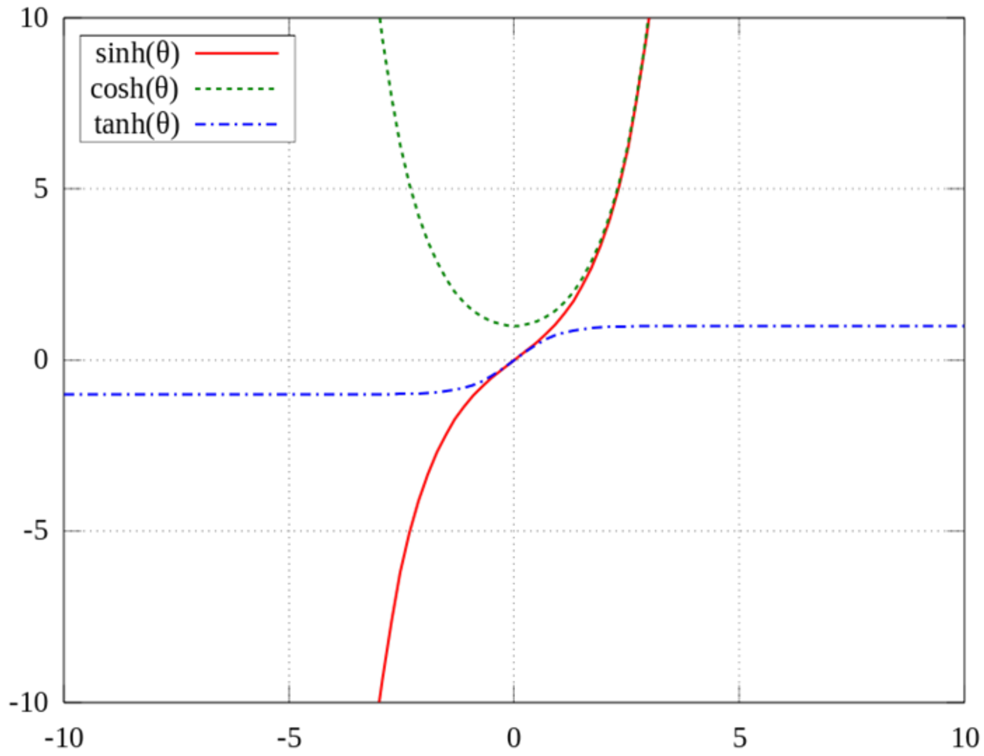


Figure: Hyperbolic Sine (red), Hyperbolic Cosine (green), and Hyperbolic Tangent (blue).

Elliptic and Hyperbolic Geometry

properties:

1. $\frac{d}{dx}\sinh x = \cosh x$; $\frac{d}{dx}\cosh x = \sinh x$
2. $\cosh^2 x - \sinh^2 x = 1$
3. Euler's Formula: $\cosh x + \sinh x = e^x$
4. $\cosh x - \sinh x = e^{-x}$
5. A homogeneous rope that is hung up at its two end points and sags only due to its own weight can be described ba a cosh-curve. Such a curve is called a **catenary**.
6. The area under the curve of $\cosh x$ is equal to its arc length:

$$A = \int_a^b \cosh(x)dx = \int_a^b \sqrt{1 + \left(\frac{d}{dx}\cosh(x)\right)^2} dx = \text{arc length}$$

Elliptic and Hyperbolic Geometry

Triangle in Hyperbolic Space

As in Euclidean space, a triangle in hyperbolic space is uniquely determined by the intersection of three lines:

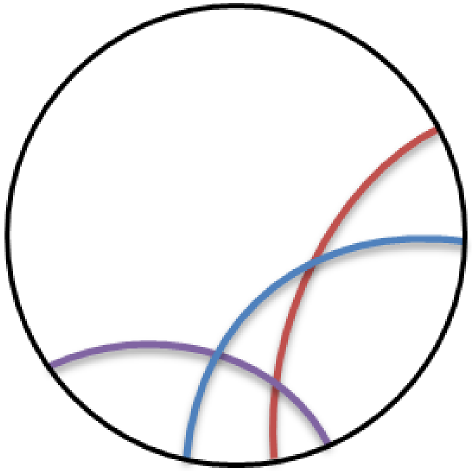


Figure: Triangle in the Poincaré disk model of the hyperbolic plane.

The sum of the angles in a triangle in hyperbolic spaces is always smaller than π, in elliptic spaces, it is always larger.

Elliptic and Hyperbolic Geometry

In hyperbolic space, the follwing theorems similar to the sine rule and the cosine rule in trigonometry hold:

Theorem

Let α, β, and γ, denote the angles of a triangle in hyperbolic space and a, b, and c the lengths of the oppisite sites. Then:

$$\frac{\sin \alpha}{\sinh a} = \frac{\sin \beta}{\sinh b} = \frac{\sin \gamma}{\sinh c}$$

$$\cosh c = \cosh a \cdot \cosh b - \sinh a \cdot \sinh b \cdot \sin \gamma$$

Elliptic and Hyperbolic Geometry

Integration into Projective Geometry We have been able to integrate affine geometry into projective geometry by the distinction of a hyperplane as absolute figure. Analogously, we can integrate the elliptic and hyperbolic geometry by the distinction of special quadrics:

Definition: Quadric

Let $f : V \times V \to K$ be a bilinear form and its corresponding quadratic form $F = V \to K;\ \vec{x} \mapsto f(\vec{x}, \vec{x})$.
Then, $Q := \{P(\vec{x}) \in P(V) | F(\vec{x}) = 0\}$ is called a 2nd-order hyperplane or **Quadric**.

Elliptic and Hyperbolic Geometry

Hyperbolic Geometry:
absolute figure: $x_0^2 + x_1^2 + \ldots + x_n^2 - x_{n+1}^2 = 0$
Quadric Q of rank $n+1$ and signature $(n; 1)$.

The points where $x_0^2 + x_1^2 + \ldots + x_n^2 < x_{n+1}^2$ are called the quadric's **inner points**.
The group of those projective transformations that fix the quadric and map inner points to inner points constitutes a subgroup of the projective group. This subgroup is isomorphic to the group of hyperbolic motions.

Elliptic and Hyperbolic Geometry

Elliptic Geometry:
absolute figure: $x_0^2 + x_1^2 + \ldots + x_n^2 + x_{n+1}^2 = 0$
Quadric Q without real points.

Elliptic geometry supports a projective incidence structure.
The group of those projective transformations that fix the absolute figure is a subgroup of the projective group. This subgroup is isomorphic to the group of elliptic motions.