Geometric Modelling Summer 2018

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http://hci.uni-kl.de/teaching/geometric-modelling-ss2018

Foundations from Analytic Geometry

What is Analytic Geometry?

Analytic Geometry

The main task of analytic geometry is to provide methods and techniques to solve geometric problems "by calculation". A suitable tool is the (coordinate independent) notion of a vector.

Vectors, Scalar Product and Vector Product

- a *vector* is given by an ordered pair of points (start and end)
- two vectors are *equal* iff. they can be constructed from one another by a parallel translation $\rightarrow$ A vector is the class of all equally directed line segments of identical length
- vectors form a *group* with respect to vector addition
- vectors from a *vector space* with respect to vector addition and scalar multiplication

This intuitive concept will now be explained formally:

Vectors, Scalar Product and Vector Product

Definition: Vector Space

A set V, on which an addition and a scalar multiplication are defined, is called a **Vector Space** on the scalar field of the real numbers, if for $\vec{a}, \vec{b}, \vec{c} \in V, \ \alpha, \beta \in \mathbb{R}$:

1. addition:
   1. $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$
   2. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
   3. $\exists \vec{0}, \quad \text{s.t. } \vec{a} + \vec{0} = \vec{a} \ \ \forall \vec{a} \in V$
   4. $\forall \vec{a} \in V : \ \exists -\vec{a}, \quad \text{s.t. } \vec{a} + (-\vec{a}) = \vec{0}$
2. scalar multiplication:
   1. $1 \cdot \vec{a} = \vec{a}$
   2. $\beta\,(\alpha \vec{a}) = (\beta \alpha)\,\vec{a}$
   3. $(\alpha + \beta)\,\vec{a} = \alpha \vec{a} + \beta \vec{a}$
   4. $\alpha(\vec{a} + \vec{b}) = \alpha \vec{a} + \alpha \vec{b}$

Vectors, Scalar Product and Vector Product

Definition: Standard Vector Space of Analytic Geometry

Let $\mathbb{R}^n$ be the set of all ordered n-tuples of real numbers, i.e.

$$\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \middle| x_i \in \mathbb{R} \right\}$$

$P \in \mathbb{R}^n$ is called a **point**.
An equivalence relation $\sim$ is introduced on
$M := \{(P, Q) \,|\, P, Q \in \mathbb{R}^n\}$ as follows:
$(P, Q) \sim (R, S) \Leftrightarrow Q_i - P_i = S_i - R_i$.
The equivalence classes on M defined by $\sim$ are called **vectors**.
This construction of equivalence classes introduces independence from the underlying coordinate system.

Vectors, Scalar Product and Vector Product

Applications:

1) parametric representation of a line:
$r = \vec{a} + t \cdot \vec{b}$

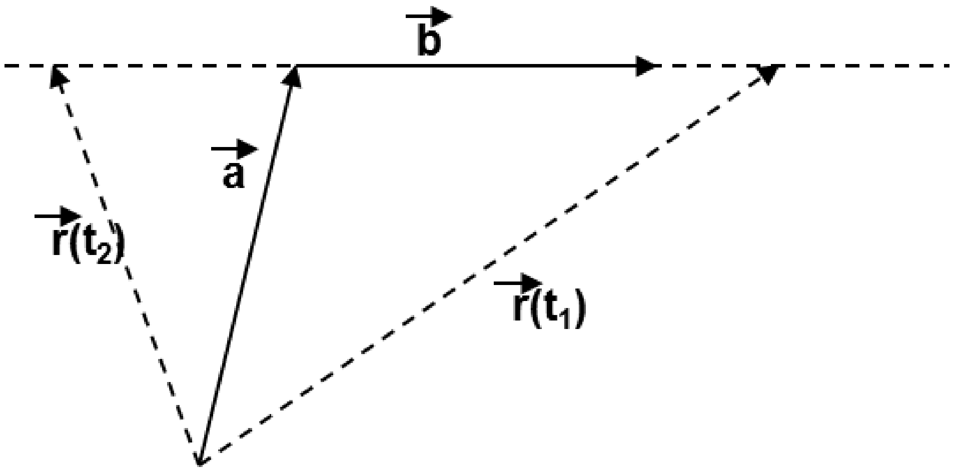


Figure: A line parameterized by a starting point $\vec{a}$ and a direction $\vec{b}$.

2) 2-point form of a line:
$r = \vec{a} + t \cdot (\vec{b} - \vec{a})$

Vectors, Scalar Product and Vector Product

Applications:

3) parametric representation of a plane:
$p = \vec{a} + t \cdot \vec{b} + \tau \vec{c}$

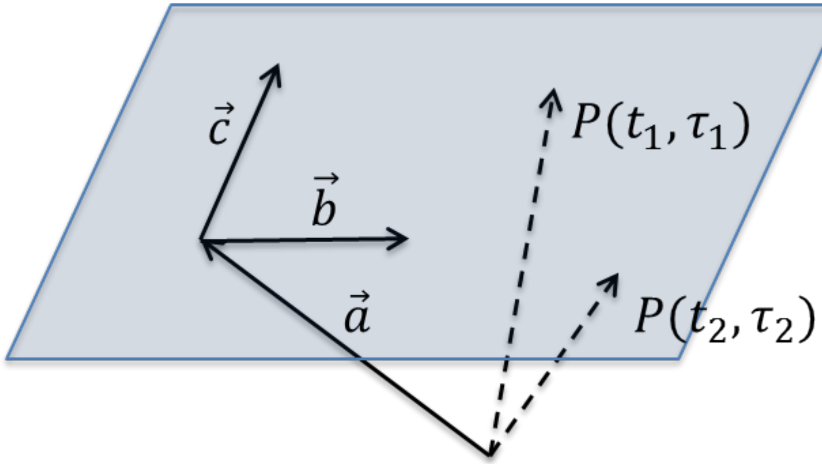


Figure: A plane parameterized by a starting point $\vec{a}$ and two directions $\vec{b}$ and $\vec{c}$.

4) 3-point form of a plane:
$p = \vec{a} + t \cdot (\vec{b} - \vec{a}) + \tau \cdot (\vec{c} - \vec{a})$

Vectors, Scalar Product and Vector Product

The following definition introduces the notion of linear (in-)dependence. This is needed to introduce suitable bases for a vector space.

Definition: Linear Dependence

n vectors $\vec{a}_1, \ldots, \vec{a}_n$ are called **linearly dependent** if there are n numbers $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ s.t. at least one of those numbers is not zero and $\alpha_1 \vec{a}_1 + \ldots + \alpha_n \vec{a}_n = \vec{0}$.
If such a set of numbers does not exist, the vectors are called **linearly independent**.

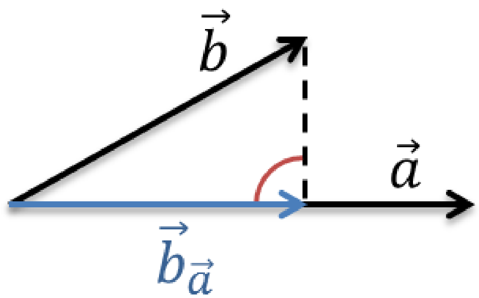
Note that a pair of two vectors are linearly dependent iff. they are parallel.

Vectors, Scalar Product and Vector Product

Definition: Scalar Product

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$$
$$\left(\vec{a}, \vec{b}\right) \longmapsto \langle \vec{a}, \vec{b} \rangle := a_1 b_1 + \ldots + a_n b_n$$

The scalar product defines a norm $\| \cdot \|$ on a vector space.
It can thus be used to introduce angles and lengths.
Generally, by defining $d(P, Q) := \|\vec{p} - \vec{q}\|$, the scalar product induces a metric on a vector space.



Vectors, Scalar Product and Vector Product

Comments:

1. The scalar product of two vectors is the multiplication of the length of the one vector times the length of the projection of the other vector onto the first one.
2. By $\|\vec{a}\| := \langle \vec{a}, \vec{a} \rangle^{1/2}$, the scalar product defines a norm $\| \cdot \| : V \to \mathbb{R}^+ \cup \{0\}$ on vector space V.
3. $\langle \vec{a}, \vec{b} \rangle = \|\vec{a}\| \cdot \|\vec{b}\| \cdot \cos \Phi$, where $\Phi := \sphericalangle(\vec{a}, \vec{b})$
4. $\langle \vec{a}, \vec{b} \rangle = 0 \Leftrightarrow \vec{a} \perp \vec{b}$

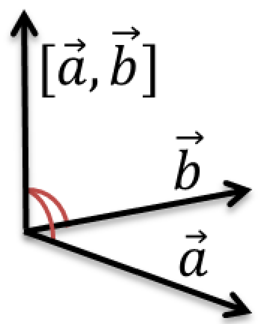
Vectors, Scalar Product and Vector Product

Definition: Vector resp. Cross Product

$$[\cdot,\cdot] : V \times V \to V; \qquad V = \mathbb{R}^3$$

$$[\vec{a},\vec{b}] \longmapsto \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} ; \{\vec{e}_1, \vec{e}_2, \vec{e}_3\} \text{ standard basis of } \mathbb{R}^3$$

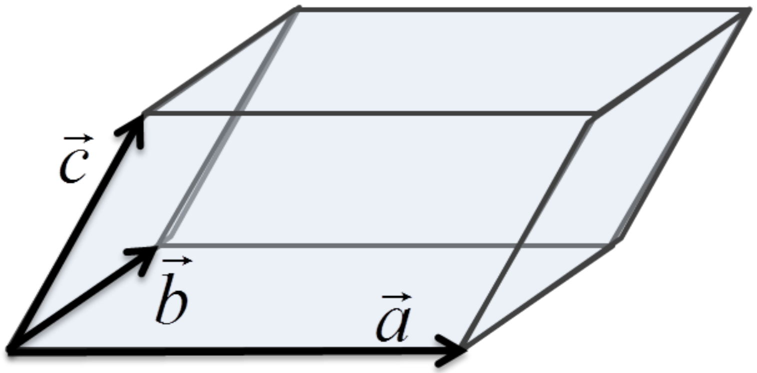
The vector product is needed to introduce the direction of normals and to define volumes.



Vectors, Scalar Product and Vector Product

Comments:

1. $[\cdot,\cdot] : V \times V \to V$ is an antisymmetric ($[\vec{a}, \vec{b}] = -[\vec{b}, \vec{a}]$), bilinear, vector valued map
2. The so-called **triple product** $\langle [\vec{a}, \vec{b}], \vec{c} \rangle$ is the (oriented) volume of the parallelepiped spanned by $\vec{a}$, $\vec{b}$, and $\vec{c}$.



Vectors, Scalar Product and Vector Product

Rules:

1. $[\vec{a}, \vec{b}] = \vec{0}$ iff. $\vec{a}, \ \vec{b}$ linearly dependent
2. $[\vec{a}, \vec{b}]$ is orthogonal to $\vec{a}$ and $\vec{b}$;
   $\{\vec{a}, \vec{b}, [\vec{a}, \vec{b}]\}$ forms a right-handed system
3. $\|[\vec{a}, \vec{b}]\| = \|\vec{a}\| \cdot \|\vec{b}\| \cdot \sin\Phi = \left(\|\vec{a}\| \cdot \|\vec{b}\| - \langle \vec{a}, \vec{b} \rangle\right)^{1/2}$
   where $\Phi := \sphericalangle(\vec{a}, \vec{b})$
4. $\langle \vec{c}, [\vec{a}, \vec{b}] \rangle = \det(\vec{c}, \vec{a}, \vec{b}) = |\vec{c}, \vec{a}, \vec{b}|$
5. $\langle [\vec{a}, \vec{b}], [\vec{c}, \vec{d}] \rangle = \langle \vec{a}, \vec{c} \rangle \langle \vec{b}, \vec{d} \rangle - \langle \vec{a}, \vec{d} \rangle \langle \vec{b}, \vec{c} \rangle$
6. $[\vec{a}, [\vec{b}, \vec{c}]] = \langle \vec{a}, \vec{c} \rangle \vec{b} - \langle \vec{a}, \vec{b} \rangle \vec{c}$
7. $[[\vec{a}, \vec{b}], [\vec{c}, \vec{d}]] = \det(\vec{a}, \vec{b}, \vec{d}) \cdot \vec{c} - \det(\vec{a}, \vec{b}, \vec{c}) \cdot \vec{d}$

Vectors, Scalar Product and Vector Product

Comments:

5) The angle of the normals of two planes can be calculated by the angles between the vectors spanning the planes (law of cosines).

6) The vector orthogonal to $\vec{a}$ and to $[\vec{b}, \vec{c}]$ lies in a plane spanned by $\vec{b}$ and $\vec{c}$. The contributions of $\vec{b}$ and $\vec{c}$ are determined by the projections of $\vec{a}$ onto $\vec{b}$ and $\vec{a}$ onto $\vec{c}$, respectively.

7) The normal of a plane spanned by the vector orthogonal to $\vec{a}$ and $\vec{b}$ and the vector orthogonal to $\vec{c}$ and $\vec{d}$ lies in a plane spanned by $\vec{c}$ and $\vec{d}$. The contributions of are determined by the respective volumes of $[\vec{a}, \vec{b}]$ and $[\vec{c}, \vec{d}]$. (7) follows from (6) by applying (4).

Vectors in Coordinate Systems

Until now, except for the definitions of the products, no coordinate systems have been involved. After defining a suitable basis (i.e. after the determination of a coordinate system), there is an unambigous assignment between (position) vectors and tuples of scalars:

$$\vec{a} = \sum_{i=1}^{n} a_i \vec{e_i}$$

$\{\vec{e_i}\}_{i=1}^{n}$ orthonormal basis

$a_i := \langle \vec{a}, \vec{e_i} \rangle$

Vectors in Coordinate Systems

"Computing" with column representations of vectors in $\mathbb{R}^3$:

$$\text{vector addition: } \vec{a} + \vec{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}$$

$$\text{scalar multiplication: } \lambda\vec{a} = \begin{pmatrix} \lambda a_1 \\ \lambda a_2 \\ \lambda a_3 \end{pmatrix}$$

Vectors in Coordinate Systems

"Computing" with column representations of vectors in $\mathbb{R}^3$:

inner/scalar product: $\langle \vec{a}, \vec{b} \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3$

outer/vector product: $[\vec{a}, \vec{b}] = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

triple product: $\langle [\vec{a}, \vec{b}] \vec{c} \rangle = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Vectors in Coordinate Systems

Applications:

1) length of vectors
2) angle between vectors
3) orientations
4) **Hesse normal form**:
   Let P_1, P_2, P_3 be three points in a plane (resp. their position vectors)

$$HF := \frac{[(P_2 - P_1), (P_3 - P_1)]}{\|(P_2 - P_1), (P_3 - P_1)\|} \longrightarrow \langle(\vec{r} - P_1), HF\rangle = 0$$

Vectors in Coordinate Systems

Replacing the "running point" $\vec{r}$ in the plane representation in the Hesse normal form by the position vector $\vec{a}$ of an arbitrary point, the distance of this point to the plane is given by $|\langle(\vec{a} - P_1), HF\rangle|$.

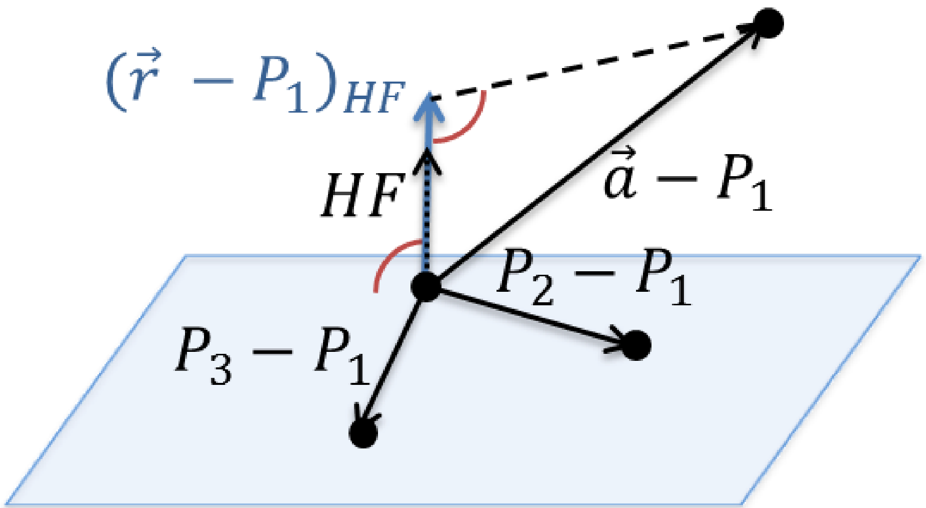


Figure: Illustration of the use of the Hesse normal form to determine the distance of a point with position vector $\vec{a}$ from a plane given by the points P_1, P_2, and P_3.

Vectors in Coordinate Systems

5) distance of a point P to a line given by $r = \vec{a} + t\vec{b}$:

$$d(r, P) = \frac{||[(P - \vec{a}), \vec{b}]||}{||\vec{b}||}$$

6) distance of the two skew lines $r = \vec{a}_1 + t\vec{b}_1$ and $s = \vec{a}_2 + \tau\vec{b}_2$:

$$d(r, s) = \frac{|\langle(\vec{a}_1 - \vec{a}_2), [\vec{b}_1, \vec{b}_2]\rangle|}{||[\vec{b}_1, \vec{b}_2]||} \text{ if } \det(\vec{a}_1 - \vec{a}_2, \vec{b}_1, \vec{b}_2) \neq 0$$

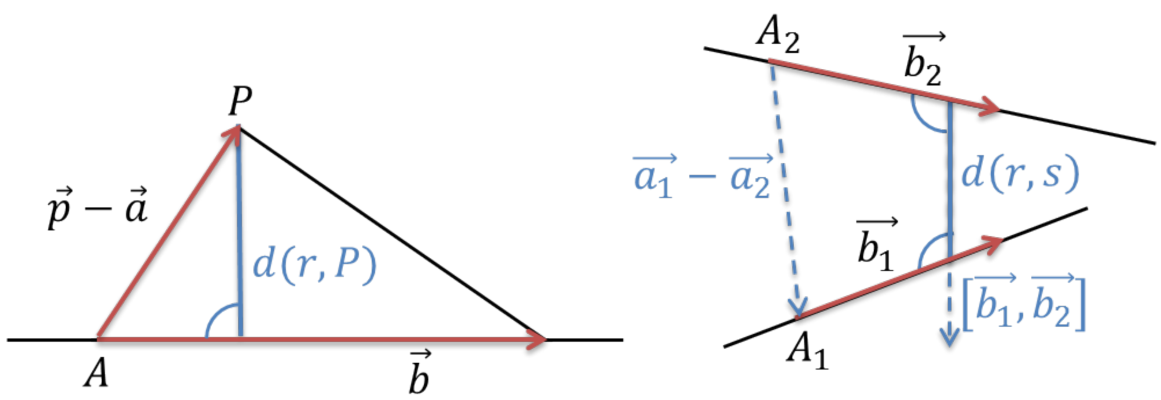


Figure: Left: Distance of point P to line r: $\frac{1}{2}d(r, P)||\vec{b}|| = \frac{1}{2}||[\vec{p} - \vec{a}, \vec{b}]||$. Right: Distance of the two skew lines r and s: $d(r, s) = ||(\vec{a}_1 - \vec{a}_2)_{[\vec{b}_1, \vec{b}_2]}||$, the projection of $(\vec{a}_1 - \vec{a}_2)$ on the normalized vector normal to r and s.

Vectors in Coordinate Systems

Positions of the respective points of shortest distance:

$$\tau_0 = \frac{|\vec{b}_1, (\vec{a}_1 - \vec{a}_2), [\vec{b}_1, \vec{b}_2]|}{\langle [\vec{b}_1, \vec{b}_2], [\vec{b}_1, \vec{b}_2] \rangle}, \quad t_0 = \frac{|\vec{b}_2, (\vec{a}_1 - \vec{a}_2), [\vec{b}_1, \vec{b}_2]|}{\langle [\vec{b}_1, \vec{b}_2], [\vec{b}_1, \vec{b}_2] \rangle}$$

These positions can of course be obtained using differential calculus but one can also proceed "geometrically":

$$d \cdot \frac{[\vec{b}_1, \vec{b}_2]}{\|[\vec{b}_1, \vec{b}_2]\|} = \vec{a}_1 - \vec{a}_2 + t_0 \vec{b}_1 - \tau_0 \vec{b}_2$$

Vectors in Coordinate Systems

Vector multiplication with $\vec{b}_1$ resp. $\vec{b}_2$ yields:

$$d \cdot \frac{[[\vec{b}_1, \vec{b}_2], \vec{b}_2]}{\|[\vec{b}_1, \vec{b}_2]\|} = [(\vec{a}_1 - \vec{a}_2), \vec{b}_2] + t_0[\vec{b}_1, \vec{b}_2]$$

resp.

$$d \cdot \frac{[\vec{b}_1, [\vec{b}_1, \vec{b}_2]]}{\|[\vec{b}_1, \vec{b}_2]\|} = [\vec{b}_1, (\vec{a}_1 - \vec{a}_2)] + \tau_0[\vec{b}_1, \vec{b}_2]$$

Vectors in Coordinate Systems

A scalar multiplication with $[\vec{b}_1, \vec{b}_2]$:

$$0 = \langle[(\vec{a}_1 - \vec{a}_2), \vec{b}_2], [\vec{b}_1, \vec{b}_2]\rangle + t_0\langle[\vec{b}_1, \vec{b}_2], [\vec{b}_1, \vec{b}_2]\rangle$$
$$0 = \langle[\vec{b}_1, (\vec{a}_1 - \vec{a}_2)], [\vec{b}_1, \vec{b}_2]\rangle + \tau_0\langle[\vec{b}_1, \vec{b}_2], [\vec{b}_1, \vec{b}_2]\rangle$$

From these, the two equations given above for t_0 and τ_0 follow. More applications can be found in K. P. Grotemeyer: *Analytische Geometrie*, a very good textbook which we widely followed in this chapter.

Higher Order Vector Spaces

Vector spaces of higher order are needed to define angles between subspaces and volumes of subspaces. This leads to an oriented vector product: The exterior product (or wedge product). The notion of the exterior product is very important for differential geometry: Applying it to infinitesimal vectors yields so-called differential forms, a coordinate-free approach to multivariate calculus. Differential forms allow for the coordinate-free integration on oriented differentiable manifolds of arbitrary dimension (curves, surfaces, volumes, ...).

Higher Order Vector Spaces

Definition: Vector Space of Order 2

Let V be an n-dim. vector space and $\{\vec{e}_1, \ldots, \vec{e}_n\}$ an orthonormal basis (ONB) of V. Then,

$$\{\vec{e}_i \wedge \vec{e}_j | i, j = 1, \ldots, n \text{ and } \vec{e}_i \wedge \vec{e}_j = -\vec{e}_j \wedge \vec{e}_i\}$$

is a basis of the second-order vector space $\Lambda^2(V)$ of dimension $\binom{n}{2}$.

Higher Order Vector Spaces

Examples and Special Cases:

1) n = 3:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \wedge \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = (a_1 \vec{e}_1 + a_2 \vec{e}_2 + a_3 \vec{e}_3) \wedge (b_1 \vec{e}_1 + b_2 \vec{e}_2 + b_3 \vec{e}_3)$$

$$\begin{aligned} = \quad & \vec{e}_1 \wedge \vec{e}_2 \,(a_1 b_2 - a_2 b_1) \\ & + \vec{e}_3 \wedge \vec{e}_1 \,(a_3 b_1 - a_1 b_3) \\ & + \vec{e}_2 \wedge \vec{e}_3 \,(a_2 b_3 - a_3 b_2) \end{aligned}$$

where $\{\vec{e}_1 \wedge \vec{e}_2, \vec{e}_3 \wedge \vec{e}_1, \vec{e}_2 \wedge \vec{e}_3\}$ is the basis of the 3-dim. 2nd-order space $\Lambda^2(\mathbb{R}^3)$.

Higher Order Vector Spaces

Examples and Special Cases:

2) in general:

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \wedge \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \left(\sum_{i=1}^{n} a_i \vec{e}_i \right) \wedge \left(\sum_{k=1}^{n} b_k \vec{e}_k \right)$$

$$= \sum_{i<k} (a_i b_k - a_k b_i) (\vec{e}_i \wedge \vec{e}_k)$$

3) the exterior product has a special meaning in $\mathbb{E}^3$:
"identify" $\vec{e}_1 \leftrightarrow \vec{e}_2 \wedge \vec{e}_3$; $\vec{e}_2 \leftrightarrow \vec{e}_3 \wedge \vec{e}_1$; $\vec{e}_3 \leftrightarrow \vec{e}_1 \wedge \vec{e}_2$.
this results in: $[\vec{a}, \vec{b}] = \vec{a} \wedge \vec{b}$.

Higher Order Vector Spaces

Examples and Special Cases:

4) from

$$\sum_{i<k} (a_i b_k - a_k b_i)^2 = \left(\sum_{i=1}^{n} (a_i)^2\right)\left(\sum_{k=1}^{n} (b_k)^2\right) - \left(\sum_{i=1}^{n} a_i b_i\right)^2$$

it follows that:

$$\begin{aligned} \|\vec{a} \wedge \vec{b}\|^2 &= \|\vec{a}\|^2 \|\vec{b}\|^2 - \langle \vec{a}, \vec{b} \rangle^2 \\ &= \|\vec{a}\|^2 \|\vec{b}\|^2 \left(1 - \frac{\langle \vec{a}, \vec{b} \rangle^2}{\|\vec{a}\|^2 \|\vec{b}\|^2}\right) \\ &= \|\vec{a}\|^2 \|\vec{b}\|^2 \left(1 - \cos^2 \Phi\right) \\ &= \|\vec{a}\|^2 \|\vec{b}\|^2 \sin^2 \Phi \end{aligned}$$

where $\Phi = \sphericalangle(\vec{a}, \vec{b})$.

Higher Order Vector Spaces

This means, $\|\vec{a} \wedge \vec{b}\| = \|\vec{a}\| \|\vec{b}\| \sin \Phi$.
Geometrically, this means that the absolute value of the exterior product of two vectors is equal to the area of the parallelogram spanned by $\vec{a}$ and $\vec{b}$.

Definition: Vector Space of Order k

Vector spaces $\Lambda^k(V)$ of order k can be defined analogously to 2nd-order vector spaces. $\Lambda^k(V)$ has the dimension $\binom{n}{k}$.
$\{\vec{a}_1, \ldots, \vec{a}_n\}$ are linearly dependent $\Leftrightarrow \vec{a}_1 \wedge \vec{a}_2 \wedge \ldots \wedge \vec{a}_k = \vec{0}$.

A special case is $\Lambda^n(V)$. This vector space is 1-dimensional and one has: $\vec{a}_1 \wedge \ldots \wedge \vec{a}_n = \det(a_{ij})(\vec{e}_1 \wedge \ldots \wedge \vec{e}_n)$.

Higher Order Vector Spaces

Theorem: "Volume Property of the Determinant"

$\|\vec{a}_1 \wedge \vec{a}_2 \wedge \ldots \wedge \vec{a}_k\|$ is the volume of the k-dim. parallelotope spanned by $\vec{a}_1$, $\vec{a}_2$, ..., $\vec{a}_k$ in $\mathbb{E}^n$ $(k < n)$:

$$\|\vec{a}_1 \wedge \ldots \wedge \vec{a}_k\| = \begin{vmatrix} \langle \vec{a}_1, \vec{a}_1 \rangle & \ldots & \langle \vec{a}_1, \vec{a}k \rangle \\ \vdots & \ddots & \vdots \\ \langle \vec{a}_k, \vec{a}_1 \rangle & \ldots & \langle \vec{a}_k, \vec{a}_k \rangle \end{vmatrix}$$

Definition: "Opening Angle" between two k-dim. Spaces

$$\sin \Phi := \frac{\|\vec{a}_1 \wedge \ldots \wedge \vec{a}_k \wedge \vec{b}_1 \wedge \ldots \wedge \vec{b}_k\|}{\|\vec{a}_1 \wedge \ldots \wedge \vec{a}_k\| \cdot \|\vec{b}_1 \wedge \ldots \wedge \vec{b}_k\|}$$

Higher Order Vector Spaces

A line is given by an equivalence class of vectors. A k-dim subspace is identifiable by an equivalence class of k-vectors as of the following theorem:

Theorem

a) For all r-dim. subspaces $U \subset V$, there is (except for scalar multiples) exactly one r-vector $\vec{e}_1 \wedge \ldots \wedge \vec{e}_r$ with $\vec{x} \in U \Leftrightarrow \vec{x} \wedge \vec{e}_1 \wedge \ldots \wedge \vec{e}_r = \vec{0}$

b) Let U_1 and U_2 subspaces of dimensions r_1 resp. r_2 and corresponding r_1-vector $\vec{w}_1$ resp. r_2-vector $\vec{w}_2$.

$$U_1 \subset U_2 \quad \Leftrightarrow \text{there is } (r_2 - r_1)\text{-vector } \vec{v} \text{ with } \vec{w}_2 = \vec{w}_1 \wedge \vec{v}$$

$$U_1 \cap U_2 = \emptyset \Leftrightarrow \vec{w}_1 \wedge \vec{w}_2 \neq \vec{0}$$

$$U_1 \cap U_2 = \emptyset \Leftrightarrow \vec{w}_1 \wedge \vec{w}_2 \text{ is } (r_1 + r_2)\text{-vector regarding } U_1 + U_2$$

Higher Order Vector Spaces

The denomination of a p-vector involves more than the definition of a subspace. Two different p-vectors that define the same oriented p-dim. subspace differ by a factor which is an invariant of the full linear (Euclidean) group. This invariant scalar can be used to define the length of a vector. In general, one obtains the notions of volume introduced above.