Geometric Modelling Summer 2018

Prof. Dr. Hans Hagen

http://hci.uni-kl.de/teaching/geometric-modelling-ss2018

Foundations from Projective Geometry

Motivating Considerations

(This follows K.P. Grotemeyer: *Analytic Geometry*.)

Starting with ordinary Cartesian (Euclidean) coordinates $\overline{x}_1$, $\overline{x}_2$, $\overline{x}_3$ of a point P in $\mathbb{R}^3$ ($\mathbb{E}^3$), we introduce four variables x_0, x_1, x_2, x_3 by carrying out the following operations:

$$\overline{x}_1 = \frac{x_1}{x_0}; \quad \overline{x}_2 = \frac{x_2}{x_0}; \quad \overline{x}_3 = \frac{x_3}{x_0}$$

These variables are called **homogeneous coordinates**.

Motivating Considerations

The (Euclidean) point P is mapped to the column vector $\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$

and vice versa every column vector of this kind with $x_0 \neq 0$ is a point in Euclidean space, but:

$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $\begin{pmatrix} cx_0 \\ cx_1 \\ cx_2 \\ cx_3 \end{pmatrix}$ map to the same point if $c \neq 0$.

Motivating Considerations

Points with $x = 0$ have no Euclidean interpretation. They are **points at infinity**. $(0, a_1, a_2, a_3)^T$ can be interpreted as the infinitely far point along the line in the direction from the coordinate offspring to the Euclidean point $(a_1, a_2, a_3)^T$.

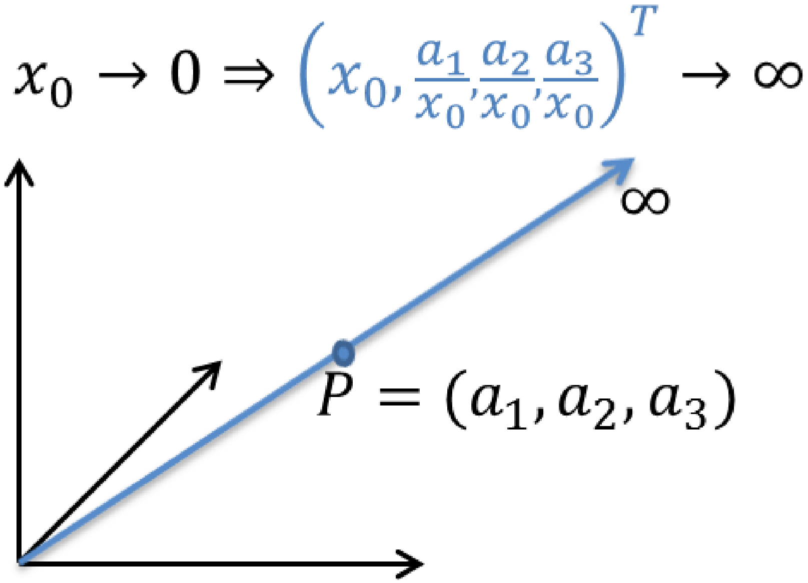


Figure: When x_0 approaches 0, the length of the direction vector approaches infinity.

Motivating Considerations

The equation $x_0 = 0$ describes the **plane at infinity**, the plane that contains all points at infinity.

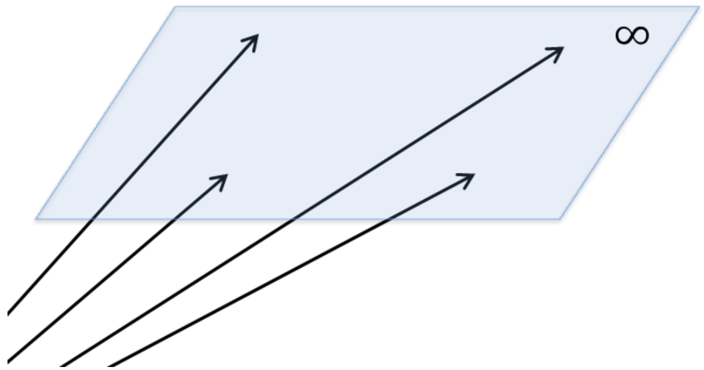


Figure: Plane at infinity: $x_0 = 0$.

$\mathbb{E}^3 +$ plane at infinity $\rightarrow P^3$ **projective space**.
In P^2, all lines intersect in one point, which is a point at infinity in case of Euclidean parallelism, or they are identical. In P^3, all planes intersect in exactly one line – probably a line at infinity – or they are identical.

Motivating Considerations

Line equation in homogeneous coordinates:

$$u_0 x_0 + u_1 x_1 + u_2 x_2 = 0$$

Plane equation in homogeneous coordinates:

$$u_0 x_0 + u_1 x_1 + u_2 x_2 + u_3 x_3 = 0$$

Motivating Considerations

Principle of Duality: Every 4-column has two geometric interpretations: As a point or as a plane.

Point	Plane
Line → connection of two points	Line → intersection of two planes
connect	intersect
line of points	sheaf of planes

Projective Transformations

The principle of duality can only be applied to results where relations between positions are concerned. In projective geometry, there are no notions of metric properties like length, angle, area, or volume!

Definition: Projective Transformations of P^3, Collineation, Correlation

Projective Transformation: $F : P^3 \to P^3$; $\tilde{\zeta} = A\zeta$; $\det A \neq 0$.
Collineation: $\tilde{\zeta}$ and ζ either both represent points or both represent planes.
Correlation: each of $\tilde{\zeta}$ and ζ may represent a point or a plane.

Projective Transformations

Definition: Linear Maps and Frames of Reference in Projective Spaces, Projective Coordinates

Let $\{\vec{\alpha}_i\}$ be linearly independent 4-columns in P^3, i.e. points or planes in P^3.

Figures of 1st order of P^3: $\zeta = \mu_0\vec{\alpha}_0 + \mu_1\vec{\alpha}_1$

Examples: lines of points, sheaf of planes, line as support of a line of points or a sheaf of planes.

Figures of 2nd order of P^3: $\zeta = \mu_0\vec{\alpha}_0 + \mu_1\vec{\alpha}_1 + \mu_2\vec{\alpha}_2$

Examples: plane as support of points (so-called bundle of planes), points as support of a bundle of planes.

Figures of 3rd order of P^3: $\zeta = \mu_0\vec{\alpha}_0 + \mu_1\vec{\alpha}_1 + \mu_2\vec{\alpha}_2 + \mu_3\vec{\alpha}_3$

Examples: all points of P^3, all planes of P^3.

μ_0, μ_1, μ_2, and μ_3 are called **projective coordinates**.

Projective Transformations

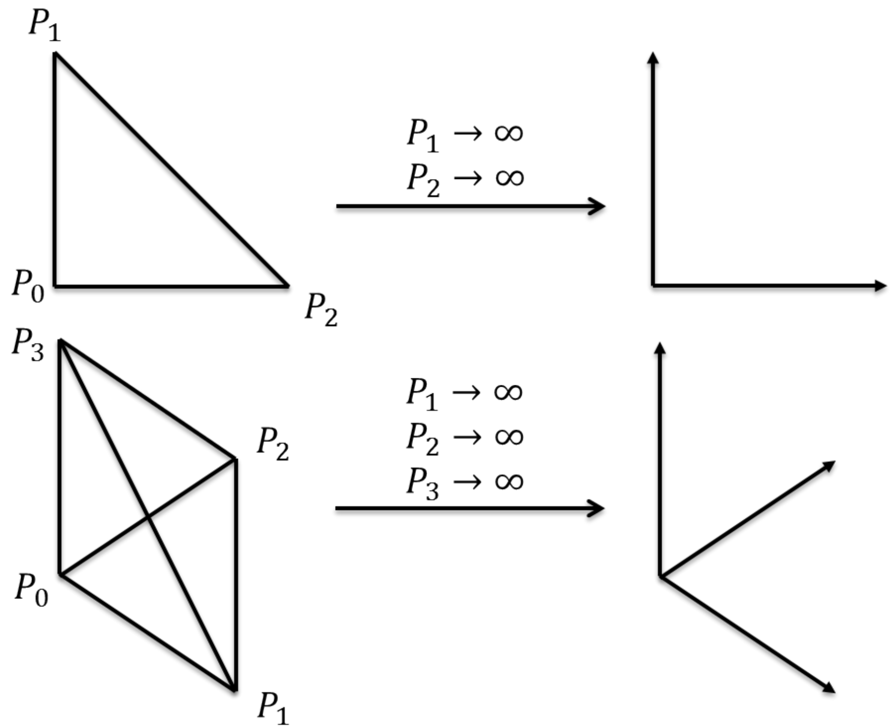
Fundamental elements of these figures:

- 1st order: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
- 2nd order: $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$; $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$; $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$
- 3rd order: $\vec{e}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$; $\vec{e}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$; $\vec{e}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$; $\vec{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

All the examples mentioned in the definition above (and more figures) can be constructed by linear projective transformations A from these fundamental elements.

Projective Transformations

In P^3, the fundamental elements form a coordinate tetrahedron.
In P^2, a coordinate triangle results.
The Cartesian (Euclidean) coordinate system features the "zero point" $\vec{e}_0$ and $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ as points at infinity.



Projective Transformations

Projective Invariants:

- projective transformations map lines to lines
- a linear figure of order r is is mapped to a figure of same order by a projective transformation. The projective coordinates stay the same
- Let ζ_1, ζ_2, ζ_3, and ζ_4 elements of a 1st-order figure and the columns $\begin{pmatrix}\alpha_0\\ \alpha_1\end{pmatrix}$, $\begin{pmatrix}\beta_0\\ \beta_1\end{pmatrix}$, $\begin{pmatrix}\gamma_0\\ \gamma_1\end{pmatrix}$, and $\begin{pmatrix}\delta_0\\ \delta_1\end{pmatrix}$ the projective coordinates of these figures. The cross ratio $CR(\zeta_0, \zeta_1, \zeta_2, \zeta_3)$ is invariant under projective transformation.

Projective Transformations

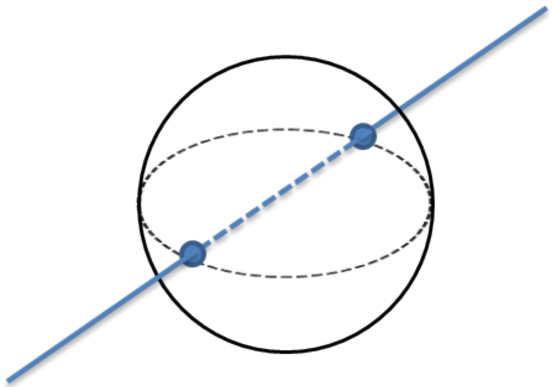
Definition: Cross Ratio

Let ζ_1, ζ_2, ζ_3, and ζ_4 elements of a 1st-order figure and the columns $\begin{pmatrix}\alpha_0\\ \alpha_1\end{pmatrix}$, $\begin{pmatrix}\beta_0\\ \beta_1\end{pmatrix}$, $\begin{pmatrix}\gamma_0\\ \gamma_1\end{pmatrix}$, and $\begin{pmatrix}\delta_0\\ \delta_1\end{pmatrix}$ the projective coordinates of these figures. Their **cross ratio** (also called double ratio) is defined as:

$$CR(\zeta_0, \zeta_1, \zeta_2, \zeta_3) = \frac{\begin{vmatrix}\alpha_0 & \gamma_0\\ \alpha_1 & \gamma_1\end{vmatrix}}{\begin{vmatrix}\beta_0 & \gamma_0\\ \beta_1 & \gamma_1\end{vmatrix}} : \frac{\begin{vmatrix}\alpha_0 & \delta_0\\ \alpha_1 & \delta_1\end{vmatrix}}{\begin{vmatrix}\beta_0 & \delta_0\\ \beta_1 & \delta_1\end{vmatrix}}$$

Projective Transformations

Two (isomorphic) models of P^2:



Points: antipodal point pairs ↔ lines through origin
Lines: great circles ↔ planes through origin

Analytical Structure of Projective Geometry

Definition: Projective Spaces

- Let V be a $(n+1)$-dim. vector space over a field F. The set of one-dim. vector spaces $\Pi K := \{t \cdot \Pi,\ t \in K,\ \Pi \in V, \Pi \neq \vec{0}\}$ is called the **projective space** of **projective dimension** n belonging to V. Notation: $P(V)$ with $\dim P(V) = n$.
- If U is a $(k+1)$-dim. subspace of V, $P(U)$ is called the k-dim **projective subspace** of $P(V)$.
  $k = 0$: "points"; $k = 1$: "lines"; $k = 2$: "planes", ...,
  $k = n - 1$: "hyperplanes"
- A point $P(\vec{r})$ is **incident** with a projective subspace $P(U)$ iff. the vector subspace $\vec{r}$ is contained in U.
- Points $P(\vec{a}_0), \ldots, P(\vec{a}_n)$ are called **linearly independent** iff. $\vec{a}_0, \ldots, \vec{a}_n$ are linearly independent vectors.

Analytical Structure of Projective Geometry

Comment:

The projective space $P(V)$ is spanned by $n+1$ linearly independent vectors but while in V after picking a base every vector can be associated with a unique $(n+1)$-tuple $(\lambda_0, \ldots, \lambda_n)$, this is not possible in $P(V)$!

An additional norming point is needed to uniquely determine the base vectors up to a common factor $p \in K \setminus \{0\}$.

Analytical Structure of Projective Geometry

Definition: Projective Coordinate Systems

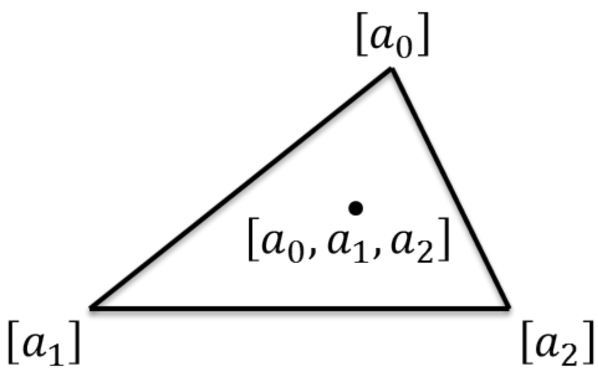
Every $(n+2)$-tuple of points of the projective space $P(V)$ having the property that $n+1$ points are always linearly independent, is called a **projective coordinate system** of $P(V)$.

Theorem

Every projective coordinate system of $P(V)$ can be represented in the form $\{P(\vec{a}_0), P(\vec{a}_1), \ldots, P(\vec{a}_n), P(\vec{a}_0 + \vec{a}_1 + \ldots + \vec{a}_n)\}$, where $\vec{a}_0, \ldots \vec{a}_n$ are a basis of V.

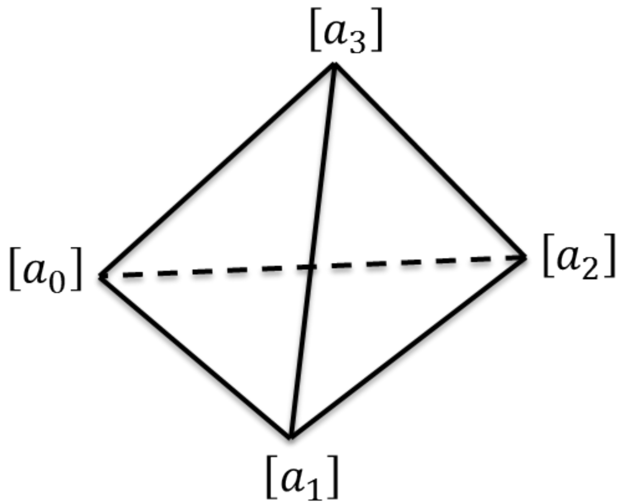
Analytical Structure of Projective Geometry

Projective Plane



coordinate triple

Projective space



coordinate quadruplet

Analytical Structure of Projective Geometry

Comments:
A k-dim. projective subspace $P(U)$ can thus be characterized by $k+1$ linearly independent points $P(\vec{a}_0), \ldots, P(\vec{a}_k)$, but also by $n-k$ linear functions $L_1, \ldots L_{n-k}$ over $V: P(U) = \{P(\vec{r}) \in P : L_1(\vec{r}) = \ldots = L_{n-k}(\vec{r}) = 0\}$.
To every k-dim. subspace of P(V) belongs a $(n-k-1)$-dim. subspace of $P(L(V,K))$. In particular, every hyperplane in $P(V)$ corresponds to a point in $P(L(V,K))$ and every point in $P(V)$ to a hyperplane in $P(L(V,K))$.

Analytical Structure of Projective Geometry

Definition: Dual Vector Spaces

Let V and V^* be vector spaces over a common scalar field F with $\dim V = \dim V^* = n+1$.

V and V^* are called **dual vector spaces** iff. there is a bilinear map $\langle \cdot, \cdot \rangle : V^* \times V \to K$ which does not degenerate, i.e.

$$\langle \Pi, x \rangle = 0 \ \forall \Pi \in V^* \quad \Rightarrow x = 0$$
$$\langle \Pi, x \rangle = 0 \ \forall x \in V \quad \Rightarrow \Pi = 0$$

Analytical Structure of Projective Geometry

Examples:

1. Let V the Euclidean vector space, $\langle \cdot, \cdot \rangle$ the scalar product.
   $\rightarrow (V, V)$ is a pair of dual vector spaces.
2. Let V the Euclidean vector space,
   $V^* := L(V, K);\ \langle f, x \rangle : f(x)$.
   $\rightarrow (V, L(V, K))$ is a pair of dual vector spaces.
3. Let $V = K^{n+1}, V^* = K^{n+1}; \Pi = (\Pi_0, \ldots \Pi_n)$,
   $x = (x_0, \ldots, x_n), \langle \Pi, x \rangle = sum_{i,j=0}^{n} c_{ij} \Pi_i x_i$.
   $\rightarrow$ If $det(c_{ij}) \neq 0$, (V, V^*) is a pair of dual vector spaces.

Analytical Structure of Projective Geometry

Definition: Projective Coordinate Systems

Let (V, V^*) be a pair of dual vector spaces. Then, $P(V^*)$ is called **dual** to $P(V)$.

Theorem

A point $P(\vec{r})$ is incident with $P(U)$ and $P(V)$ iff. $\langle \Pi, \vec{r} \rangle = 0$ for all $\Pi \in U^+ = \{\Pi \in V^* : \langle \Pi, \vec{a} \rangle = 0 \ \forall \vec{a} \in V\}$.

Theorem: Duality Principle

If one takes a theorem holding in $P(V)$ and replaces every occurence of "connection" by "intersection", "intersection" by "connection", and "k-dim. subspace" by "$(n-k-1)$-dim subspace, one obtains a new "dual theorem" that holds in $P(V)$.

Analytical Structure of Projective Geometry

Definition: Automorphism

Let V a vector space with $\dim V = n+1$. Every invertible linear map $A : V \to V$ is called automorphism of V. The set of all automorphisms of V constitute a group called $GL(V)$ resp. $GL(n+1, K)$.
After picking a base for V, every automorphism A of V bijectively associated with a quadratic matrix with non-zero determinant.

Analytical Structure of Projective Geometry

Definition: Group of Projective Transformations

If $A \in GL(n+1, K)$, the map $\alpha : P(V) \to P(V)$ induced by A on $P(V)$ is called a **projective transformation**. The automorphisms A and A' of V induce the same map α on $P(V)$ if $A = c \cdot A'$ for some $c \neq 0$.
The set of all homothetic transformations $H_{n+1} := \{c \cdot Id | c \neq 0\}$ is a normal subgroup (dt. "Normalteiler") of $GL(n+1, K)$. The quotient group (dt. "Faktorgruppe")

$$\mathbb{P} := P.GL(n+1, K) := GL(n+1, K)/H_{n+1}$$

is the **group of the projective transformations** of P(V).

The next chapter will discuss further detail on the group theoretical aspects within the scope of Felix Klein's Erlanger Programm.

Analytical Structure of Projective Geometry

Properties of Projective Transformations:

- bijective
- preserve the cross ratio
- map projective subspaces to projective subspaces of equal dimension

Analytical Structure of Projective Geometry

Properties of Projective Transformations:

- bijective
- preserve the cross ratio
- map projective subspaces to projective subspaces of equal dimension

Analytical Structure of Projective Geometry

Comments:

- Every line-preserving bijective map that preserves the cross ratio from $P(V)$ to itself is a projective transformation.
- The subject of projective geometry is to investigate properties of configurations and figures that are invariant under projective transformations.
- Every line-preserving bijective map of a real projective space whose dimension is at least two is a projective transformation.