Geometric Modelling Summer 2018

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Foundations from Vector Calculus

Foundations from Vector Calculus

In this chapter, we briefly introduce some fundamental concepts from multivariate calculus. These concepts are important for geometry in general and differential geometry in particular.

We start with differential calculus:

Multivariate Differential Calculus

Definition: Partial Derivative

Let $M \subset \mathbb{R}^n$ open; $F : M \rightarrow \mathbb{R}$
The **partial derivative** at point $a = (a_1, a_2, \ldots, a_n) \in M$ with respect to the i-th variable a_i is defined as:

$$\frac{\partial}{\partial x_i} F(a) = \lim_{h \Rightarrow 0} \frac{F(a_1, \ldots, a_i + h, \ldots, a_n) - F(a_1, \ldots, a_i, \ldots, a_n)}{h}$$

Interpreting this as the directional derivative in direction $\vec{e}_i$, we can define the **directional derivative in direction** $\vec{a}$ as:

$$\lim_{t \rightarrow 0} \frac{F(t_0 + t_{\vec{a}}) - F(x_0)}{t}$$

Multivariate Differential Calculus

Further, we define the **Nabla operator** in $\mathbb{R}^n$ as: $\nabla = \sum_{i=1}^{n} \vec{e}_i \frac{\partial}{\partial i}$, where $\vec{e}_i$ is the unit vector in i-direction.

Thus, if $F : M \subset \mathbb{R}^n \to \mathbb{R}$, we can define the **gradient** of F as: $\mathrm{grad} F = \nabla F = \left(\frac{\partial}{\partial x_1} F, \dots, \frac{\partial}{\partial x_n} F \right)$

Multivariate Differential Calculus

Clairot's Theorem (dt. meist: Satz von Schwarz):

Let $M \subset \mathbb{R}^n$ open; $F : M \to \mathbb{R}$; F at least k times differentiable and all k-th derivatives continuous in M. Then, the order of the sequence of derivations in all j-th derivatives with $j \leq k$ is irrelevant.

Especially for the second derivative, we get:

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}F\right) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}F\right)$$

In this course, we will also use different notations, like:

$$\frac{\partial^2 F}{\partial x \partial y}(x, y) = \frac{\partial^2 F}{\partial y \partial x}(x, y) \text{ or } F_{xy} = F_{yx}$$

Multivariate Differential Calculus

Now, we define the notion of differentiability for mulvariate real functions mapping to arbitrary real spaces:

Definition: Differentiable Function

Let $M \subset \mathbb{R}^n$ open; $F : M \to \mathbb{R}^m$; $x_0 \in M$.
F is called **differentiable** in a point $a \in M$, iff. there exists a linear map

$$dF_a : \mathbb{R}^n \to \mathbb{R}^m \ \text{ s.t. } \ \lim_{\|h\| \to 0} \frac{F(a+h) - F(a) - dF_a(h)}{\|h\|} = 0$$

Note that h is a vector!

Multivariate Differential Calculus

A function F is differentiable iff. there is a map dF_a as defined above. dF_a is a unique map and called the **differential of F in a**. With respect to the standard bases $\mathbb{R}^n$ and $\mathbb{R}^m$, it is given by the Jacobian matrix:

Definition: Jacobian Matrix

Let $F : \mathbb{R}^n \to \mathbb{R}^m$ and $\frac{\partial F}{\partial x_i}$ be defined for all $i = 0, \ldots, n$. Then, the **Jacobian Matrix** of F at point a is defined as:

$$J_F(a) = J \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} := \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(a) & \ldots & \frac{\partial f_1}{\partial x_n}(a) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(a) & \ldots & \frac{\partial f_m}{\partial x_n}(a) \end{bmatrix}$$

If F differentiable, $dF_a = J_F(a)$.

Multivariate Differential Calculus

Remarks:

- The Jacobian of a scalar-valued univariate function is the derivative.
- The Jacobian of a scalar-valued multivariate function is the gradient.
- Intuitively, the Jacobian describes the local “amount of transforming” that is imposed by a transformation.
- If a function is differentiable, its differential is given by the Jacobian. The converse is in general not true!
  The Jacobian only requires the partial derivatives of a function to exist. Thus, in general, the Jacobian can be defined for a function that is not differentiable.

Multivariate Differential Calculus

Remarks:

- If $F : M \to \mathbb{R}^m$ has **continuous** partial derivatives, dF_a is defined.
- If $m = 1$, the gradient of F points to the direction of the greatest increment and its length is equal to the greatest increment.
- Generalization of the chain rule: $d(g \circ f)|_a = dg|_{f(a)} \cdot df|_a$
In coordinates: $J(g \circ f)|_a = J\left(g\left(f(a)\right)\right) \cdot Jf(a)$

Example:

$$x\left(u(t), v(t), w(t)\right) \longrightarrow \frac{\partial x}{\partial t} = x_u \cdot \dot{u} + x_v \cdot \dot{v} + x_w \cdot \dot{w}$$

Multivariate Differential Calculus

Implicit Function Theorem

Let $M \subset \mathbb{R}^n$ open; $(x_0, y_0) \in M$; $F : M \to \mathbb{R}^k$ a $\mathcal{C}^r$-continuous map with $F(x_0, y_0) = 0$ and the differential of $y \mapsto F(x_0, y_0)$ be regular in y_0.
Then, there exists a subspace $V \subset \mathbb{R}^(n-k)$ and a $\mathcal{C}^r$-continuous map $G : V \to \mathbb{R}^k$ where $G(x_0) = y_0$ and $F\,(x, G(x)) = 0$ for all $x \in V$.

Multivariate Differential Calculus

The following is a useful corollary that can be derived from the implicit function theorem:

Inverse Function Theorem

Let the differential of a function $F : U \to V$, where $U, V \subset \mathbb{R}^n$, be regular, i.e. F is differentiable and $\det J_F \neq 0$ (and thereby J_F invertible).
Then, there exists a local inverse map $\left(F|_u\right)^{-1} : V \to M; \quad M \subseteq U$.

Multivariate Integrals

Definition: Line Integral (Curve Integral, Contour Integral)

Let $V : M \to \mathbb{R}^n$ a continuouos vector field and K a piecewise smooth oriented curve in $\mathbb{R}^n$ with parameterization $F : [a, b] \to \mathbb{R}^n$. The number

$$\int_K \langle V, dF \rangle := \int_a^b \langle V\left(F(t)\right), F'(t) \rangle dt$$

is independent from the parameterization and is called the **line integral** (also: curve integral or contour integral) **of** V **along** K.

Multivariate Integrals

Substitution Rule:
Let $K \subset \mathbb{R}^n$ compact; f integrable on K.
By $g : \mathbb{R}^n \to \mathbb{R}^n$, "new coordinates" are introduced:

$$\int_K f(\vec{x}) d\vec{x} = \int_{g^{-1}(K)} f\left(g(\tilde{\vec{x}}) \det(g')\right)$$

$\det(g')$ is some kind of "distortion factor".

Multivariate Integrals

Example Coordinate Systems:

- **Polar Coordinates in the Plane**:

$$g: \begin{array}{l} x = r\cos\varphi \\ y = r\sin\varphi \end{array} \quad \text{with domain:}$$

$$D(g) = \{(r,\varphi) | r \in [0,\infty); \varphi \in [0,2\pi]\}$$

- **Spherical Coordinates in Space**:

$$g: \begin{array}{l} x = r\cos\varphi\cos\theta \\ y = r\sin\varphi\sin\theta \\ z = r\sin\theta \end{array} \quad \text{with domain:}$$

$$D(g) = \{(r,\varphi,\theta) | r \in [0,\infty); \varphi \in [0,2\pi]; \theta \in [-\tfrac{\pi}{2},\tfrac{\pi}{2}]\}$$

- **Cylindrical Coordinates in Space**:

$$g: \begin{array}{l} x = r\cos\varphi \\ y = r\sin\varphi \\ z = \tilde{z} \end{array} \quad \text{with domain:}$$

$$D(g) = \{(r,\varphi,\tilde{z}) | r \in [0,\infty); \varphi \in [0,2\pi]; \tilde{z} \in (-\infty,\infty)\}$$

Multivariate Integrals

Examples:

1) $\int_K \sqrt{x^2+y^2}d(x,y)$ where $K := \{(x,y)|1 \le x^2+y^2 \le 4\}$:

new coordinates: polar coordinates in plane: $\begin{matrix} x = r\cos\varphi \\ y = r\sin\varphi \end{matrix}$

$$g^{-1}K = \{(r,\varphi)|1 \le r \le 2; 0 \le \varphi \le 2\pi\}$$

$$\int_K \sqrt{x^2+y^2}d(x,y) = \int_{g^{-1}(K)} r\det(g')d\tilde{x}$$

$$= \int_{g^{-1}(K)} r^2 d(r,\varphi) = \int_0^{2\pi}\int_1^2 r^2 dr d\varphi$$

$$= \frac{14}{3}\pi$$

Multivariate Integrals

Examples:

2) **Volume of the Sphere Octant**:

$V = \int_K d(x,y,z)$ where

$K = \{(x,y,z) | x \geq 0; y \geq 0; x^2 + y^2 + z^2 \geq 1\}$

new coordinates: spherical coordinates:
$$\begin{aligned} x &= r\cos\varphi\cos\theta \\ y &= r\sin\varphi\sin\theta \\ z &= r\sin\theta \end{aligned}$$

$$g^{-1}(K) = \{(r,\varphi,\theta) | 0 \leq r \leq 1; 0 \leq \varphi \leq \tfrac{\pi}{2}; 0 \leq \theta \leq \tfrac{\pi}{2}\}$$

$$\det(g') = r^2\cos\theta$$

$$v = \iiint_K d(x,y,z) = \iiint_{g^{-1}(K)} r^2\cos\theta d\varphi d\theta dr$$

$$= \int_0^1 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} r^2\cos\theta d\varphi d\theta dr = \frac{\pi}{6}$$

Multivariate Integrals

Applications: **Centroids**:

$$\begin{bmatrix} x_1 \\ y_0 \\ z_0 \end{bmatrix} = \frac{1}{V} \cdot \begin{bmatrix} \iiint x \ d(x,y,z) \\ \iiint y \ d(x,y,z) \\ \iiint z \ d(x,y,z) \end{bmatrix} ; \quad V := \iiint d(x,y,z)$$

Multivariate Integrals

Applications: **Centroids**:

Example: Centrois of the sphere octant:

$$x_0 = \frac{6}{\pi}\iiint_K x \; d(x,y,z) = \frac{6}{\pi}\iiint_{g^{-1}(K)} r^3 \cos\varphi \cos^2\theta \; dr d\varphi d\theta$$
$$= \frac{6}{\pi}\int_0^1 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} r^3 \cos\varphi \cos^2\theta \; dr d\varphi d\theta$$
$$= \frac{3}{8}$$

Analoguously, one gets $y_0 = \frac{3}{8}$ and $z_0 = \frac{3}{8}$.

Multivariate Integrals

Applications: **Moments of Inertia**:
Moments of inertia of a volume V with respect to the x-axis (I_x), y-axis (I_y), and z-axis (I_z):

$$I_x := \iiint_V (y^2 + z^2)\ d(x,y,z)$$

$$I_y := \iiint_V (x^2 + z^2)\ d(x,y,z)$$

$$I_z := \iiint_V (x^2 + y^2)\ d(x,y,z)$$

Multivariate Integrals

Applications: **Moments of Inertia**: Example: Moment of inertia of a cuboid:

$$\begin{aligned} V =& \{(x,y,z) | x \in [-\tfrac{a}{2}, \tfrac{a}{2}], y \in [-\tfrac{b}{2}, \tfrac{b}{2}], z \in [-\tfrac{c}{2}, \tfrac{c}{2}]\} \\ I_x =& \iiint_V (y^2 + z^2)\ d(x,y,z) \\ =& \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{c}{2}}^{\frac{c}{2}} (y^2 + z^2)\ dxdydz \\ =& \frac{a \cdot b \cdot c}{12} (b^2 + c^2) \end{aligned}$$

I_y and I_z can be computed analogously.

Multivariate Integrals

Important Vector Fields: remember: $\nabla = \vec{e}_1 \frac{\partial}{\partial x_1} + \vec{e}_2 \frac{\partial}{\partial x_2} + \cdots + \vec{e}_n \frac{\partial}{\partial x_n}$

- $\operatorname{grad} f(x) = \nabla f := \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right),$
  where $f : \mathbb{R}^n \to \mathbb{R};\ f \in \mathcal{C}^1$
- $\operatorname{div} v(x) = \langle \nabla, v \rangle := \frac{\partial V_1}{\partial x_1}(x) + \frac{\partial V_2}{\partial x_2}(x) + \cdots + \frac{\partial V_n}{\partial x_n}(x),$
  where $v : \mathbb{R}^n \to T \subseteq \mathbb{R}^n;\ v \in \mathcal{C}^1$
- $\operatorname{curl} v(x) = [\nabla, v] := \left(\frac{\partial V_3}{\partial x_2} - \frac{\partial V_2}{\partial x_3}, \frac{\partial V_1}{\partial x_3} - \frac{\partial V_3}{\partial x_1}, \frac{\partial V_2}{\partial x_1} - \frac{\partial V_1}{\partial x_2} \right),$
  where $v : \mathbb{R}^3 \to T \subseteq \mathbb{R}^3;\ v \in \mathcal{C}^1$

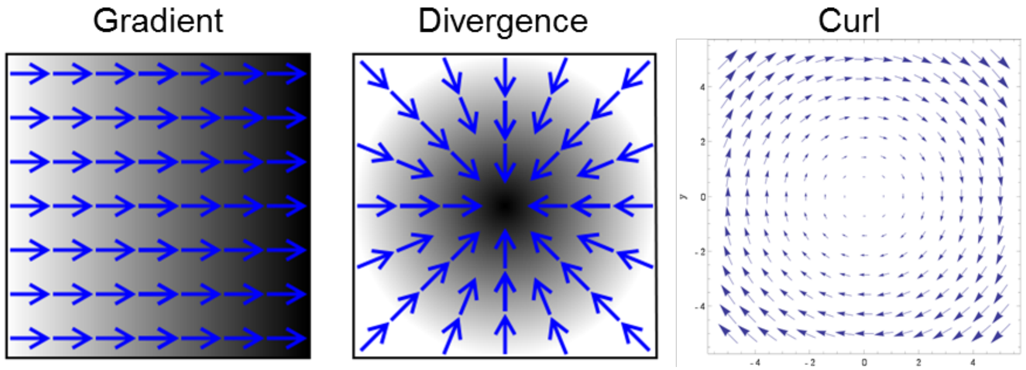


Figure: Gradient, Divergence, and Curl. Images taken from Wikipedia.

Multivariate Integrals

Definition: Surface Integrals

Let $X(u)$ be a surface. The area of the surface is defined as:

$$\iint_{X(u)} dF := \iint_{u} \|[\tfrac{\partial X}{\partial u_1}(u_1, u_2), \tfrac{\partial X}{\partial u_2}(u_1, u_2)]\| \; d(u_1, u_2)$$
$$= \iint_{u} \sqrt{g} \; du_1 du_2$$

If $G : u \to R$, the **surface integral of** G **over** $X(u)$ is defined as:

$$\iint_{X(u)} G \; dF := \iint_{u} G(u_1, u_2) \cdot \sqrt{g} \; du_1 du_2$$

Often, one has $G(u, v) = \langle F \circ X, N \rangle$, where $F : \mathbb{R}^3 \to \mathbb{R}^3$ and N is the surface normal.

Multivariate Integrals

Definition: Flux of a Vector Field

The **flux** of a vector field $A : \mathbb{R}^3 \to T \subseteq \mathbb{R}^3$ through a surface $X(u)$ with surface normal N is defined as:

$$\begin{aligned}\iint_{X(u)} \langle A \circ X, N \rangle \; dF &= \iint_{u} \langle A \circ X, N \rangle \| [X_1, X_2] \| \; d(u, v) \\ &= \iint_{u} \langle A \circ X, N \cdot \| [X_1, X_2] \| \rangle \; d(u, v) \\ &= \iint_{u} \langle A \circ X, [X_1, X_2] \rangle \; d(u, v)\end{aligned}$$

Multivariate Integrals

Remarks:

The flux can be used to model transport: For example, the *energy flux* measures the rate of energy that passes an oriented unit area (e.g. heat flux, radiation flux). Another example is the *particle flux*, the number of particles per second that pass a unit area.

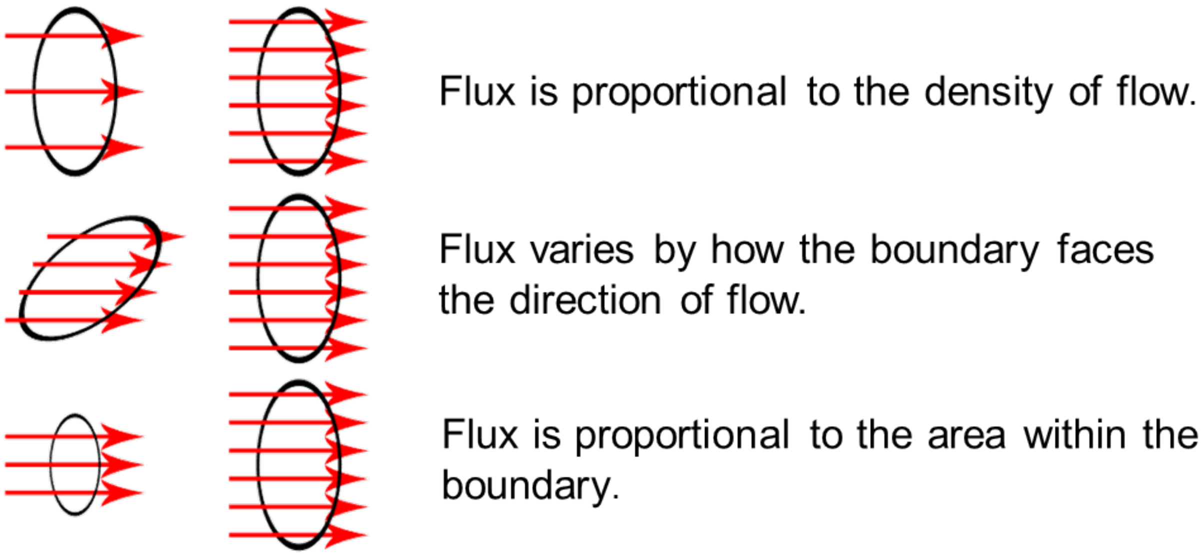


Figure: Red Arrows: Flow of particles, charges, etc. Black circles: Surface boundaries. The flux is the number of arrows passing each ring. Text and images taken from Wikipedia.

Integral Theorems

Green's Theorem

Let $\mathbb{V}$ a vector field on $D \subset \mathbb{R}^2$, V be some region $\subset D$, and ∂V the piecewise smooth boundary curve of V. Furthermore, let v_1, v_2 continuous functions $V \to \mathbb{R}$. Then:

$$\iint_V \left(\frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1} \right) d(x,y) = -\int_{\partial V} (v_1 dx_1 + v_2 dx_2)$$
$$= -\int_{\partial V} \langle \vec{v}, d\vec{x} \rangle$$

Integral Theorems

Divergence Theorem (Gauss' Theorem)

Let $\mathbb{V}$ be a vector field over $D \subset \mathbb{R}^3$, V some region $\subset \mathbb{V}$, v a continuous differentiable vector field over an open set U with $V \subseteq U$. Furthermore, let ∂V the outer surface of V in $\mathbb{R}^3$ and N the outside normal of said surface. Then:

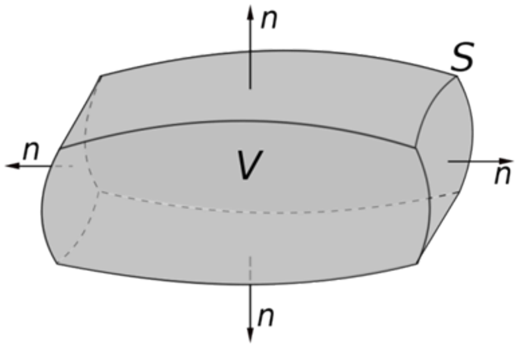
$$\iiint_V \operatorname{div} v \; d(x, y, z) = \iint_{\partial V} \langle v, N \rangle dS$$

Statement: "The flux of v through the surface ∂V of V is equal to the integral of the source density over V."

$\int_{\partial V}$ is the surface integral formed with the outer surface element dS on ∂V.

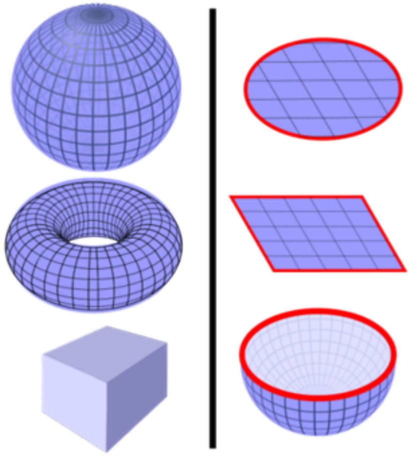
Integral Theorems

Remarks:



A closed region V in space with boundary surface $S = \partial V$ and outer surface normal n.

The Divergence Theorem can be directly applied to the calculation of the flux through a fully enclosed volume.



The Divergence Theorem is **not** directly applicable to surfaces with boundaries!

Figure: Upper: Closed region in space. Lower: Examples for surfaces, where the Divergence Theorem is applicable (left) and where it is not (right). Boundaries in red. Images taken from Wikipedia.

Integral Theorems

Stokes' Theorem

Let $\mathbb{V}$ be a vector field over $D \subset \mathbb{R}^3$, v be a continuous differentiable vector field, X a surface in $\mathbb{R}^3$ with piecewise smooth boundary with surface normal N. Then:

$$\iiint_{X(u)} \langle (\mathrm{curl} v) \circ X, N \rangle dF = \int_{\partial X(u)} v_1 dx_1 + v_2 dx_2 + v_3 dx_3$$
$$= \int_{\partial X(u)} \langle v, d\vec{x} \rangle$$

Statement: “The circulation of the $\mathcal{C}^2$-field v along $\partial X(u)$ is equal to the flux of curl$v$ through $X(u)$.”

Coordinate-Free Representation of grad, div, and curl

Let $\mathcal{V}$ a region of space with volume V and f a function continuous around $p \in \mathcal{V}$. The following function is sometimes called a "volume integral":

$$f(p) = \lim_{V \to 0} \frac{1}{V} \int_{\mathcal{V}} f(x)dx$$

Using this equation and certain special cases of the divergence theorem, we can derive coordinate-free representation of the gradient, the divergence, and the curl:

Coordinate-Free Representation of grad, div, and curl

Let $\mathcal{V}$ a spatial region of volume V, $\partial\mathcal{V}$ the boundary surface of $\mathcal{V}$.

- The **gradient** of the scalar field f in a point $p \in \mathcal{V}$ is given by

$$\operatorname{grad} f(p) = \lim_{V \to 0} \frac{1}{V} \int_{\partial\mathcal{V}} f \; d\vec{S}$$

- The **divergence** of a vector field v in a point $p \in \mathcal{V}$ is given by

$$\operatorname{div} v(p) = \lim_{V \to 0} \frac{1}{V} \int_{\partial\mathcal{V}} v \; d\vec{S}$$

- The **curl** of a vector field v in a point $p \in \mathcal{V}$ is given by

$$\operatorname{curl} v(p) = \lim_{V \to 0} \frac{1}{V} \int_{\partial\mathcal{V}} [d\vec{S}, v]$$

$\int_{\partial\mathcal{V}}$ is the surface integral formed with the outer surface element dS on $\partial\mathcal{V}$.