Geometric Modelling Summer 2018

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http://hci.uni-kl.de/teaching/geometric-modelling-ss2018

Differential Geometry
Curve Theory

Differential Geometry – Curve Theory

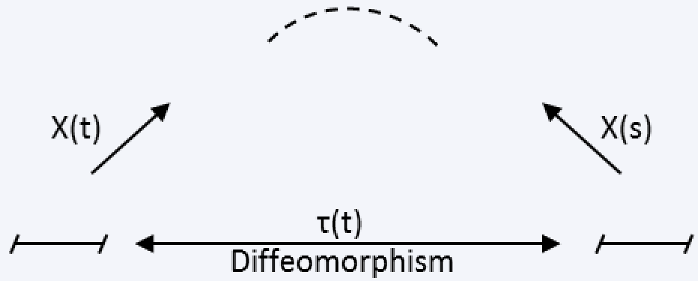
The aim of differential-geometric methods is to describe geometric properties by means of the differential calculus. These properties usually conern local characteristics. Combined with topological methods, there also arise several interesting global results.

Parametric Curves

Definition (Parametric Curves – Concept of a Curve)

a) A **parameterised curve** of class $\mathcal{C}^r$ in $\mathbb{R}^n$ is a map $X : [a, b] \to \mathbb{R}^n$ of class $\mathcal{C}^r (r \geq 1)$. The curve is called regular if $X(t)$ has maximal degree on the whole interval $[a, b]$ $(\Rightarrow X'(t) \neq 0)$.

b) A curve in $\mathbb{R}^n$ is an **equivalence class of parametric curves** wherein a diffeomorphism $\tau : [a, b] \to [\bar{a}, \bar{b}]$ of class $\mathcal{C}^r$ is called **parameter transformation** of class $\mathcal{C}^r$.
(τ bijective, r times continuously differentiable, $\tau'(t) \neq 0$)

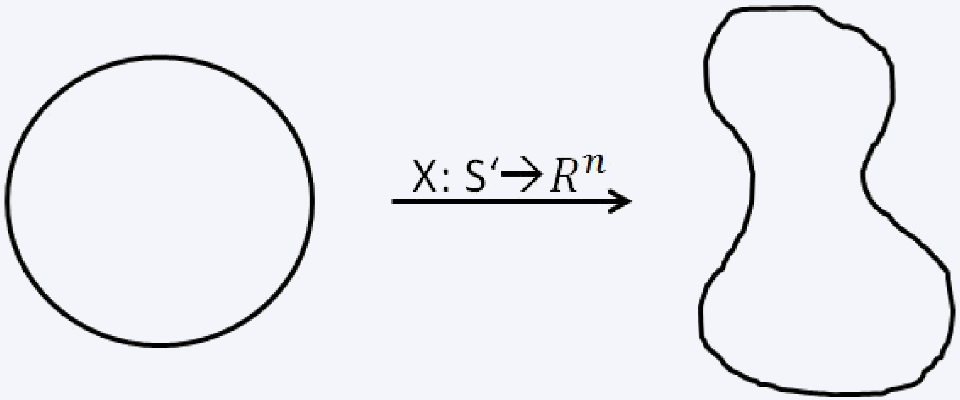


„Reparametrization“

Parametric Curves

Definition cont'd (Parametric Curves – Concept of a Curve)

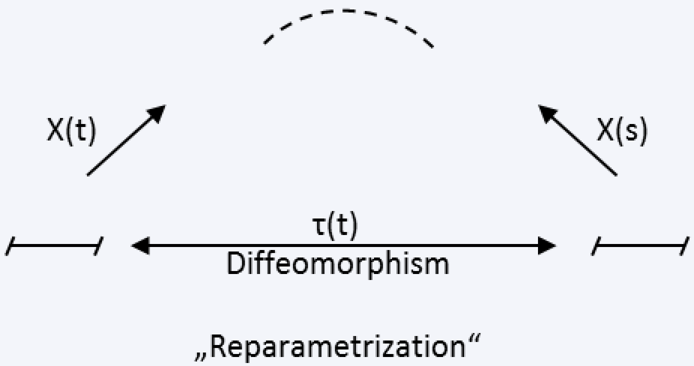
- c) A **closed curve** of class $\mathcal{C}^r$ in $\mathbb{R}^n$ is a map of the circle line in $\mathbb{R}^n$, $X : S' \to \mathbb{R}^n$ $\quad (S' := \{(\cos\alpha, \sin\alpha), 0 \leq \alpha \leq 2\pi\})$ with the property: for each interval $[a, b]$ the map $\bar{X} : [a, b] \to \mathbb{R}^3$ defined by $\bar{X}(\alpha) := X((\cos\alpha, \sin\alpha))$ $\alpha \in [a, b]$ is a parametric curve of class $\mathcal{C}^r$.
  The curve is called simple (no double points) if X is injective.



Parametric Curves

Definition (Arc Length Parametrization)

Let K be a regular curve of class C^1 in $\mathbb{R}^n$ with length L. Then there exists a regular parametric representation $X : [0, L] \to \mathbb{R}^n$ with $\|X'(s)\| = 1$ for all $s \in [0, L]$. This is called **arc length parametrization**.



There exists a special reparametrization, that sets all tanget vectors of the curve to 1.

Moving Frames of Reference

Principle of the Moving Frame of Reference

Frames of References are plain tools. One can fix the Frame or let it "move with the curve" and describe the curve and its properties with the changing of the moved frame.

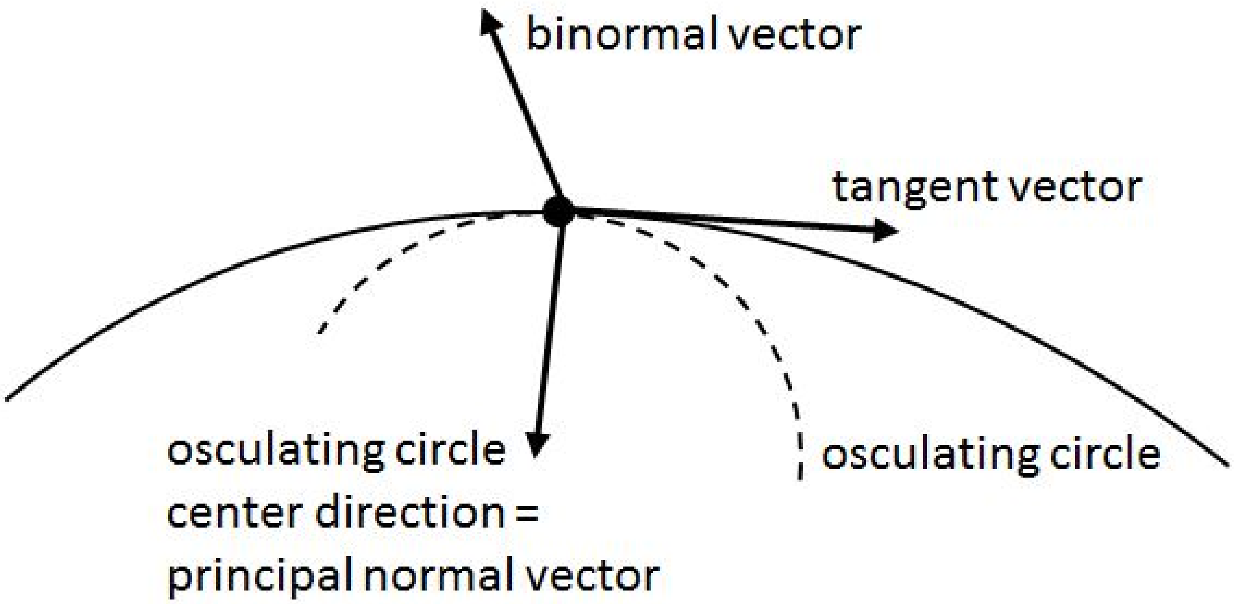


Figure: Frenet frame

Moving Frames of Reference

Frenet Frame with Arc Length Parametrization:

i. e. $\|\dot{x}\| = 1$
and $\|\ddot{x}\| \neq 0$

$V_1 := \dot{x}$ tangent vector

$V_2 := \dfrac{\ddot{x}}{\|\ddot{x}\|}$ principal normal vector

$V_3 := [V_1, V_2]$ binormal vector

The tangent vector can be interpreted as a velocity vector and the principal normal vector as a “curvature vector”.

Moving Frames of Reference

Frenet Frame with Arbitrary Parametrization:

$$e_1 := \frac{\dot{x}}{\|\dot{x}\|} \qquad \text{tangent vector}$$

$$e_2 := \frac{[\dot{x}, [\dot{x}, \ddot{x}]]}{\|\dot{x}\| \| [\dot{x}, \ddot{x}] \|} \qquad \text{principal normal vector}$$

$$e_3 := \frac{[\dot{x}, \ddot{x}]}{\| [\dot{x}, \ddot{x}] \|} \qquad \text{binormal vector}$$

Moving Frames of Reference

Curvature Vector:
In several Computer Aided Geometric Design applications, the so-called

$$\text{“curvature vector”} \quad \kappa \cdot e_2 = \frac{[\dot{x}, [\dot{x}, \ddot{x}]]}{\|\dot{x}\|^4}$$

plays an important role.

$\{V_1, V_2, V_3\}$, $\{e_1, e_2, e_3\}$ respectively, form a moving (orthonormal) frame of reference. Hence, so-called derivation rules apply. These describe the infinitesimal movement of the frame of reference. With this “movement”, one can describe the torsion and the curvature. Curvature is the deviation from the straight run, torsion is the “extrication” from the plane.

Moving Frames of Reference

Frenet Equations for Space Curves

With arc length parametrization, we have $\|\dot{x}\| = 1$

$$
\begin{aligned}
\dot{e}_1 &= \|\dot{x}\| \cdot \kappa \cdot e_2 \\
\dot{e}_2 &= -\|\dot{x}\| \cdot \kappa \cdot e_1 + \|\dot{x}\| \cdot \tau \cdot e_3 \\
\dot{e}_3 &= -\|\dot{x}\| \cdot \tau \cdot e_2
\end{aligned}
$$

matrix notation: $$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = \begin{bmatrix} 0 & \|\dot{x}\| \cdot \kappa & 0 \\ -\|\dot{x}\| \cdot \kappa & 0 & \|\dot{x}\| \cdot \tau \\ 0 & \|\dot{x}\| \cdot \tau & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

curvature $\kappa(t) := \frac{\|[\dot{x}(t), \ddot{x}(t)]\|}{\|\dot{x}(t)\|^3}$ torsion $\tau(t) := \frac{\det(\dot{x}(t), \ddot{x}(t), \dddot{x}(t))}{\|[\dot{x}(t), \ddot{x}(t)]\|^2}$

Moving Frames of Reference

Frenet Equations for Planar Curves:

$$\tau \equiv 0 \qquad \kappa(t) := \frac{\det(\dot{x},\ddot{x})}{\|\dot{x}\|^3}$$

Planar Curves

$$\begin{aligned} \dot{e}_1 &= && \|\dot{x}\| \cdot \kappa \cdot e_2 \\ \dot{e}_2 &= -\|\dot{x}\| \cdot \kappa \cdot e_1 && \end{aligned}$$

matrix notation: $$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \end{bmatrix} = \begin{bmatrix} 0 & \|\dot{x}\| \cdot \kappa \\ -\|\dot{x}\| \cdot \kappa & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

Moving Frames of Reference

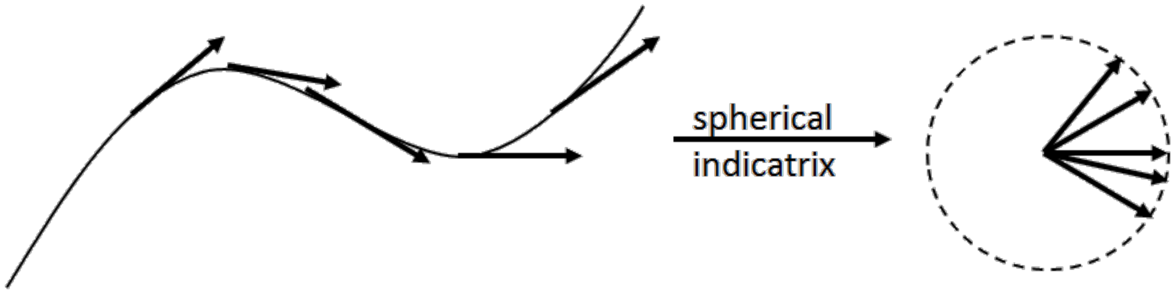
Remarks:

1. The torsion is a measure of the deviation of a curve from a planar path; the curvature is a measure of the deviation from a straight-lined path.
2. A curve is planar iff its torsion vanishes.

Moving Frames of Reference

Remarks:

(3) The map $V_1 : [0, L] \to S^2 \subset \mathbb{R}^3$ is called spherical indicatrix of tangents and $\int_0^L \kappa(s)ds$ is called **total curvature** of a curve. For closed curves, $\int_0^L \kappa(s)ds \geq 2\pi$ holds as the spherical indicatrix of tangents crosses every great circle on S^2.



(4) For $\kappa(s_0) \neq 0$ and $\tau(s_0) \neq 0$, there exists exactly one sphere, a so-called osculating sphere, touching $X(s)$ in s_0 at third degree.

Fundamental Theorem of Curve Theory

Fundamental Theorem of Curve Theory

For functions $\kappa(s)$ and $\tau(s)$, with $\kappa : [0, L] \to \mathbb{R}$ continuously differentiable and $\kappa > 0$, as well as $\tau : [0, L] \to \mathbb{R}$ with τ continuous, there exists a curve and with the exception of motion exactly one curve with $\kappa(s)$ as curvature, $\tau(s)$ as torsion, and s as arc length parameter.

Curvature and torsion create a so-called complete system of invariants, i. e. independent from the reference system, they describe the curve completely except for its position and orientation in space! Each invariant of a curve in $\mathbb{E}^3$ is a functional of curvature and torsion.
Example: The elasticity of a bent rod (if the material does not break) is just a functional of curvature and torsion.

Fundamental Theorem of Curve Theory

Global Result for Planar Curves:

Four-vertex theorem (Vierscheitelsatz)

Each regular, convex, closed planar curve has at least 4 vertices (Scheitelpunkte) (i. e. $\kappa' = 0$ for at least 4 points).

Example: Ellipse, circle with constant curvature only has vertices (Scheitelpunkte).

Fundamental Theorem of Curve Theory

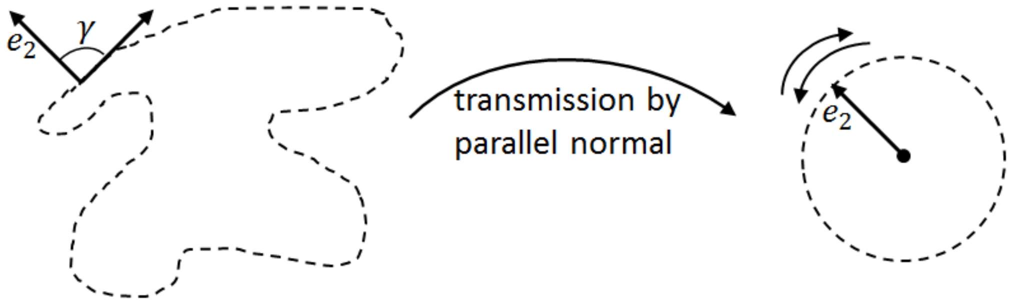
Remarks:

1. There exist closed simple (no double points) space curves that have only two points with $\kappa' = 0$.
2. The total curvature is a topological invariant as the **Theorem of Fenchel** ($\int_0^L \kappa(s)ds \geq 2\pi$) has an essential generalization: the more the curve is knotted within the meaning of knot theory, the greater the total curvature. In particular, the total curvature is $\geq 4\pi$ if the curve is not topologically equivalent to a circle.
   Example: Total curvature of a “Bretzel Curve” is 4π.

Fundamental Theorem of Curve Theory

Remarks:

4. **Concept of the Winding Number**:



$n_x := \frac{1}{2\pi} \int_0^L d\gamma$, wherein $\gamma' = \frac{d\gamma}{dt} = \kappa(t) \cdot \|X'(t)\|$

The winding number n_x of a regular, closed curve $X(t)$ is an integer and one gets: $n_x = \frac{1}{2\pi} \int \kappa(t) \cdot \|X'(t)\| dt$.

n_x is invariant under orientation-preserving reparametrizations and direct (orientation preserving) motions.

For a common simple-closed curve (topologically equivalent to a circle), one obtains $n_x = \pm 1$.

Fundamental Theorem of Curve Theory

Remarks:

5. **Darboux Vector**:

$$D(t) := \tau(t)e_1(t) + \kappa(t)e_3(t)$$

$D(t)$ is the rotation vector of the Frenet system with angular velocity $\omega(t) = \sqrt{\kappa^2(t) + \tau^2(t)}$.