Geometric Modelling Summer 2018

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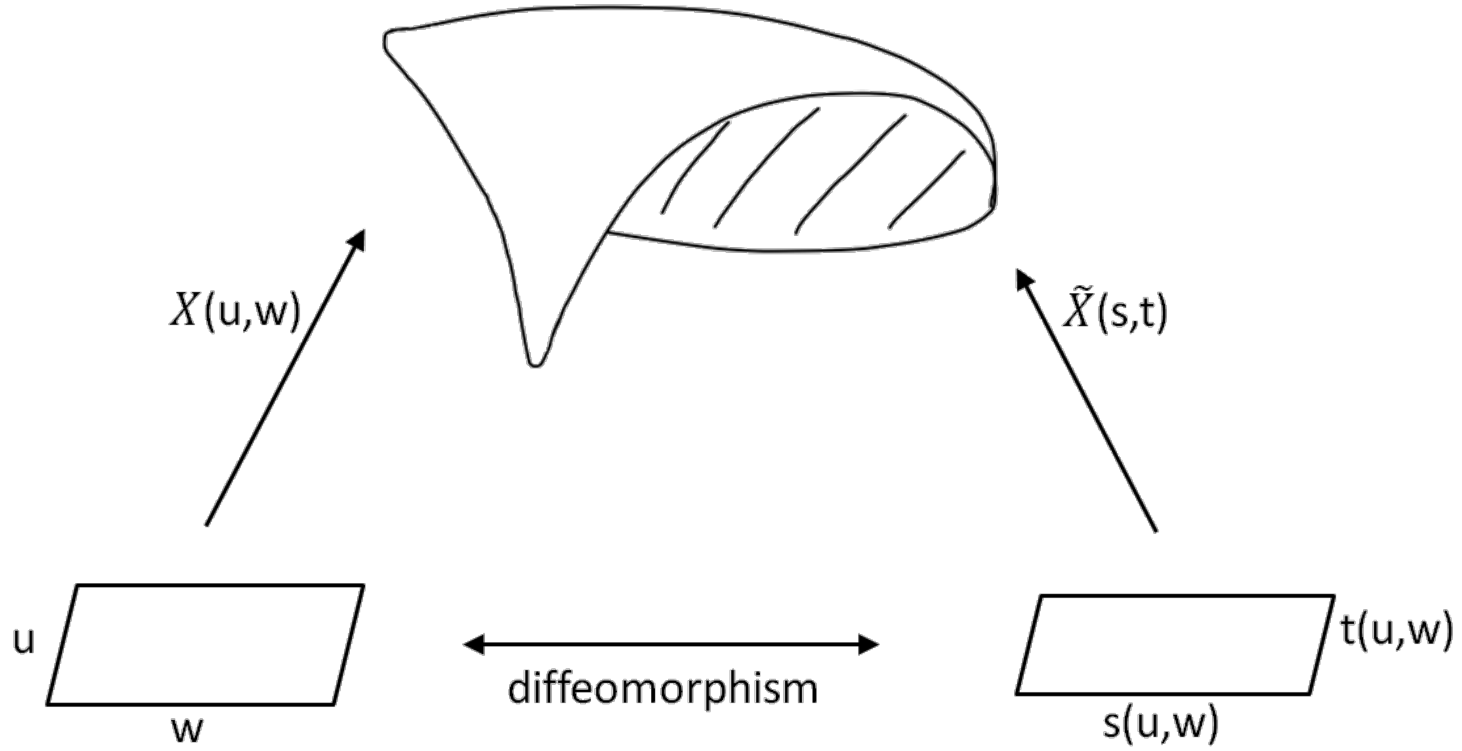
http://hci.uni-kl.de/teaching/geometric-modelling-ss2018

Differential Geometry
Surface Theory

Surface Theory



„reparametrization of the surface“

Surface Definition

Definition (Parametrized Surface)

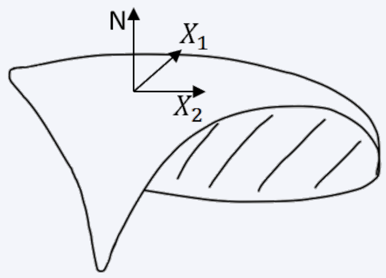
- (a) A *parametrized surface* of class $\mathcal{C}^r$ $(r \geq 1)$ is a map $X : U \to \mathbb{R}^n$ that has rank 2 everywhere (i. e. $X1, X2$ are linearly independent; $X_i := \frac{\partial X}{\partial U_i}$).
- (b) An *(oriented) surface* is an equivalence class of parametrized surfaces. Inside a surface, a diffeomorphism $\tau : \bar{U} \to U$ of class $\mathcal{C}^r$ with a Jacobian that is positive everywhere is called a parameter transformation of class $\mathcal{C}^r$.

Definition: Tangent Space, Normal Vector

Definition (Tangent Space – Normal Vector – Frame of Reference)

a) The linear subspace of $\mathbb{R}^n$ spanned by X_1, X_2 is called *tangent space* of X.

b) $N := \frac{[X_1, X_2]}{\|\sim\|}$ is called *normal unit vector* and $\{X_1, X_2, N\}$ form a (non-orthonormal) moving frame of reference called *Gauss frame*.



Surface Metric

As a motivation, consider a surface curve

$$y : [a, b] \to U \subset \mathbb{R}^2$$

$$\text{planar curve} \mapsto X : U \to \mathbb{E}^3$$

$$\text{surface} \mapsto \text{surface curve } \tilde{X}(t) = X \circ Y(t) : [a, b] \to \mathbb{E}^3$$

Calculating the length of a surface curve

$$L = \int_a^b \|\dot{\tilde{X}}\| dt = \int_a^b \left\langle \sum_{r=1}^{2} X_r \cdot \dot{U}_r, \sum_{s=1}^{2} X_s \cdot \dot{U}_s \right\rangle^{\frac{1}{2}} dt$$

$$= \int_a^b \left(\underbrace{\langle X_1, X_1 \rangle}_{g_{11}} \dot{U}_1 \dot{U}_1 + 2 \underbrace{\langle X_1, X_2 \rangle}_{g_{12}} \dot{U}_1 \dot{U}_2 + \underbrace{\langle X_2, X_2 \rangle}_{g_{22}} \dot{U}_2 \dot{U}_2 \right)^{\frac{1}{2}} dt$$

leads to the definition on the next slide...

First Fundamental Form

Definition (First Fundamental Form)

The map I assigning for each $u \in U$ the restriction of the scalar product $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ to the tangent space in U is called first fundamental form of the parametrized surface.
$G_{ij} := \langle X_i, X_j \rangle$ are the coordinates of I with respect to the canonical basis X_1, X_2 in tangent space.

First Fundamental Form Remarks

1) The tangent space is invariant under parameter transformations.
2) The first fundamental form is invariant under Euclidean motion and parameter transformation; the matrix of the first fundamental form w. r. t. the canonical basis is symmetric and positive definite.

First Fundamental Form Remarks

Knowing the first fundamental form we can determine:

- **Length of surface curves**:
  $L = \int_a^b \|\tilde{X}'(t)\| dt = \int_a^b \sqrt{g_{rs} \dot{U}^r \dot{U}^s} dt$
  (we changed the notation and sum over all indices that are written as sub- and superscripts)
- **Angle between two surface curves**: $\cos r = \left\langle \frac{\tilde{X}'(t_0)}{\|\sim\|}, \frac{\bar{X}'(t_1)}{\|\sim\|} \right\rangle$
- **surface area**: $A = \iint_U g^{\frac{1}{2}} du^1 du^2$; wherein $g := g_{11} g_{22} - g_{12}^2$

Second Fundamental Form

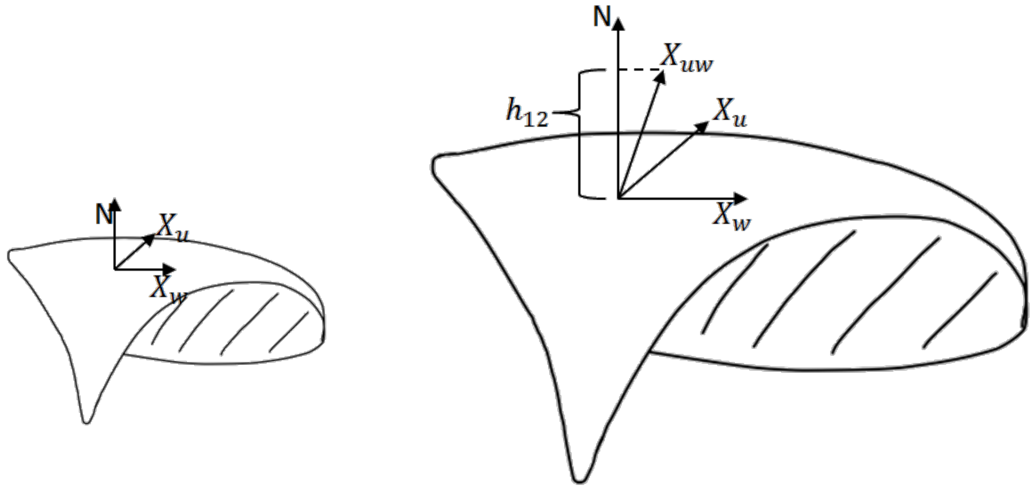
Definition (Second Fundamental Form)

The linear map $L_U : T_U X \to T_U X$ with $L_U := -dN_U \circ X_U^{-1}$ is called **shape operator** (dt. Weingartenabbildung). The bilinear form II_U defined by

$$\mathrm{II}_U(A, B) = \langle L_U(A), B \rangle \qquad \forall A, B \in T_U X$$

is called **second fundamental form** of X in U. The map $U \mapsto \mathrm{II}_U$ is called **second fundamental form of the surface**.

Second Fundamental Form - Graphical Description



Derivations of X_u in u-direction, X_w in w-direction, X_u in w-direction and X_w in u-direction lead to the vectors X_{uu}, X_{ww}, X_{uw} and X_{wu}. These can be projected to the normal N. The resulting functions h_{11}, h_{22}, h_{12} and h_{21} provide information about the curvature of the surface.

Second Fundamental Form

Theorem

1. *The shape operator is self-adjoint w. r. t. the first fundamental form.*
2. *The second fundamental form is symmetric.*
3. *The matrix* $\{h_{ij}\}_{i=1}^{2}$ *of the second fundamental form is given by:*
$$h_{ij} = -\langle X_i, N_j \rangle = +\langle X_{ij}, N \rangle \,.$$
4. *Matrix of the shape operator:* $\left(h_j^i\right) = h_{jk} g^{ki}$.

Second Fundamental Form

Weingarten Equations (dt. Weingart'sche Ableitungsgleichungen)

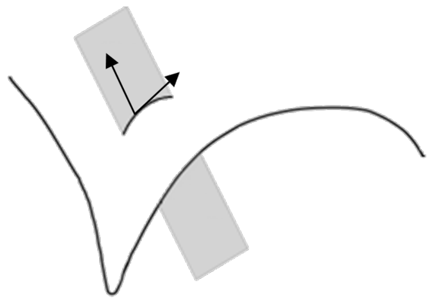
$$N_i = -h_i^r X_r$$

(Note that the partial derivative in u and in w direction of the normal vector lies in tangent space.)

Second Fundamental Form Remarks

1) The second fundamental form is invariant under direct Euclidean motion and orientation-preserving parameter transformations.
2) As L is a self-adjoint map, it has two real eigenvalues k_1 and k_2 which are called **principal curvature**. The corresponding eigenvectors are orthogonal and are called principal curvature vectors.

Algebraic facts in geometric meaning:



Look at the plane spanned by N and the direction of any tangent vector in a certain point on the surface and intersect the plane with said surface. The local sectional curve obtained lies on the surface and is a 2-dimensional curve in the intersecting normal plane. This is guaranteed by the existance of a normal vector in every point on the surface. Now rotate the tangents around N. The result is a one-parametric set of section curves (normal-section-curves). All of these planar curves are surface curves with a well-defined curvature (normal-section.curvature). The maximum and minimum values of these curvatures are exactly the eigenvalues of the Shape Operator.

Gaussian Curvature and Mean Curvature

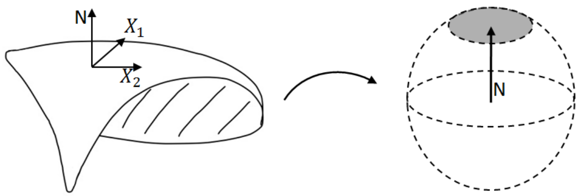
From the two principle curvatures, derive more curvature concepts:

Definition (Gaussian Curvature and Mean Curvature)

1. Gaussian curvature: $K = k_1 \cdot k_2 = \det(L) = \frac{\det \mathrm{II}}{\det \mathrm{I}}$
2. Mean curvature: $H = \frac{1}{2} trace(L) = \frac{1}{2}(k_1 + k_2)$

Now, studying curves on surfaces provides insight into the geometric meaning of the fundamental forms.

Geometric Interpretation of K



$X : U \to \mathbb{E}^3$ par. surface
$N : U \to S^2 \subset \mathbb{E}^3$ spherical map
F^*: surface area of "normal surface"

$$F^* = \iint_U |N_1, N_2, N| du^1 du^2 = \iint_U |k| g^{\frac{1}{2}} du^1 du^2 = \iint_F |k| dF$$

Now, we consider a sequence of neighborhoods of U contracting on U:

$$\lim \frac{\iint_{U_n} |k| dF}{\iint_{U_n} dF} = |K(U)|$$

The gaussian curvature can be used to classify surfaces. This is described further down in detail.

Curvature of Surface Curves

Let $Y(s)$ be an arc length parameterized surface curve

$$Y'' = k \cdot e_2 = k_n \cdot N + k_g \mathrm{S}; \quad \text{wherein } \mathrm{S} \in T_U X$$

Y'' points into the direction of the Frenet-Frame's principal normal e_2 while the surface curve still has a normal and a tangential component w. r. t. the surface. The normal component k_n is the *normal curvature* (outer geometry), the tangential component k_g is the *geodesic curvature* (inner geometry).

In the general case, e_2 (curves) and N (surface) embrace an angle φ that changes for each point. This angle depends on the curve and the surface:

$$\cos\varphi = \langle e_2, N \rangle \rightarrow \underbrace{k}_{\textit{curve}} \cdot \cos\varphi = \langle k \cdot e_2, N \rangle = \langle Y'', N \rangle = \underbrace{k_n}_{\textit{surface}}$$

Curvature of Surface Curves

Theorem (Normal Section Curvature – Meusnier – Euler)

1. *All surface curves through $X(U)$ with the same tangent in this point have the same normal curvature k_n in U. The normal curvature is given by*

$$k_n = \frac{h_{ij}\lambda^i\lambda^j}{g_{ij}\lambda^i\lambda^j}; \quad \lambda^i X_i \text{ tangent vector.}$$

2. *For φ the angle between the curve principal normal vector e_2 and the surface normal vector N the curvature k, k_n respectively, is given by*

$$k_n = k \cdot \cos\varphi \qquad \text{Meusnier's Theorem}$$

Curvature of Surface Curves

Theorem – cont'd

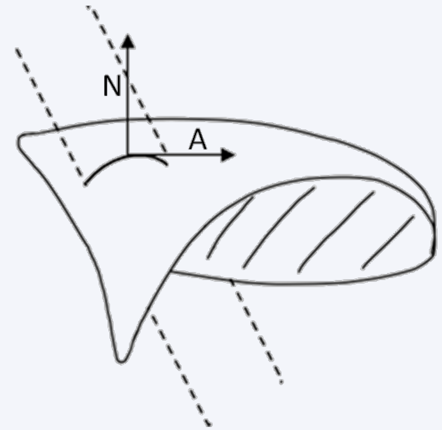
3. For $A = \lambda^i X_i \neq 0$ let $\mathfrak{X}(\lambda^1, \lambda^2)$ be the curvature of the (appropriately parameterized) sectional curve of $X(u, w)$ with the plane through $X(u, w)$ spanned by A and N. It is called **normal section curvature** of the surface in $X(u, w)$ in direction A. One has:

$$\mathfrak{X}(\lambda^1, \lambda^2) = \frac{h_{ij}\lambda^i \lambda^j}{g_{ij}\lambda^i \lambda^j}$$

.

Curvature of Surface Curves

Theorem – cont'd



$Y(s)$ is called **normal section curve** with $Y'(s_0) = A$ and $e_2(s_0) = \pm N$; the curve's existence (and hence the existence of the normal section curvature) is a result of the theorem on implicitly defined functions.

Curvature of Surface Curves

Theorem – cont'd

4. If the curvature $\mathfrak{X}$ is constant, $\mathrm{II}_U = k \cdot \mathrm{I}_U$ and $X(u_0, w_0)$ is called **umbilical point** (dt. Nabelpunkt). Otherwise, the curvature has exactly two critical values, k_1 and k_2, i. e. exactly the principal curvatures. The corresponding unit vectors A_1 and A_2 are principal curvature vectors. They are orthogonal. If $A = \lambda^i X_i = \cos\varphi A_1 + \sin\varphi A_2$, one has:

$$\mathfrak{X}(\lambda^1, \lambda^2) = k_1 \cos^2\varphi + k_2 \sin^2\varphi \qquad \textit{Euler's Theorem.}$$

Motivation for the Conception of Mean Curvature

Mean Curvature

The mean curvature is closely related to te concept of minimal surfaces.

$$H = \frac{1}{2\pi}\int_0^{2\pi} \mathfrak{X}(\varphi)d\varphi = k_1\frac{1}{2\pi}\int_0^{2\pi}\cos^2\varphi d\varphi + k_2\frac{1}{2\pi}\int_0^{2\pi}\sin^2\varphi d\varphi$$
$$= \frac{1}{2}(k_1 + k_2)$$

Motivation for the Conception of Mean Curvature

The fact that maximal and minimal curvature define orthogonal tangent directions allows curvature-line networks.

Definition (Line of Curvature)

A surface curve for which in every point the tangent vector equals the principal curvature vector is called **line of curvature**.

Representation of the Surface

Local Representation of the Surface

$X : U \to \mathbb{E}^3$ param. surface;

$f(\eta^1, \eta^2)$ distance of the point $X(u_0^1 + \eta^1, u_0^2 + \eta^2)$ to the tangential plane

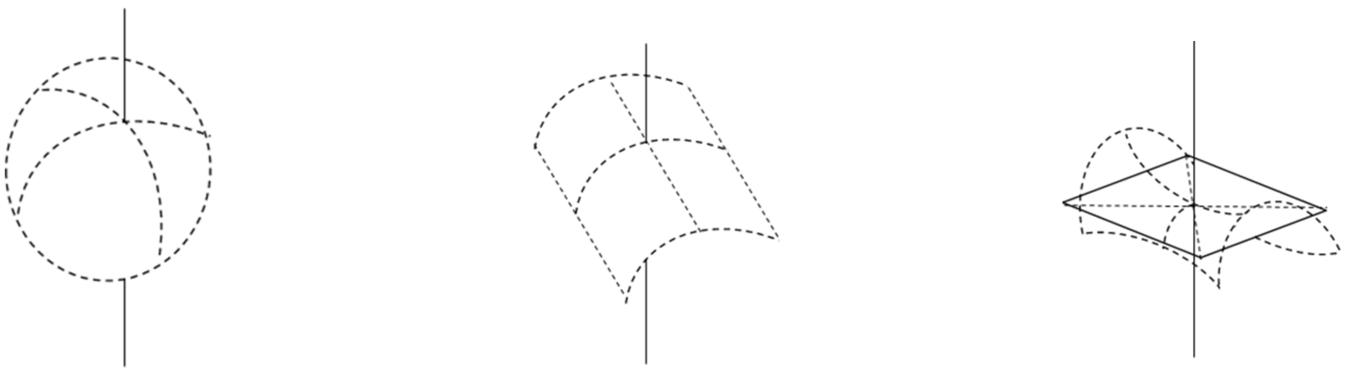
$$X(u_0^1 + \eta^1, u_0^2 + \eta^2)$$
$$= X(u_0^1, u_0^2) + X_i(u_0^1, u_0^2)\eta^i + \frac{1}{2}X_{ij}(u_0^1, u_0^2)\eta^i\eta^j + \mathcal{O}(\|\eta\|^2)$$

$$F(\eta^1, \eta^2) = \langle X(u_0 + \eta) - X(u_0), N(u_0)\rangle = \frac{1}{2}h_{ij}(u_0)\eta^i\eta^j + \mathcal{O}(\|\eta\|^2)$$

Definition (Osculating Paraboloid)

$$P(\eta^1, \eta^2) = X(u_0) + X_i(u_0)\eta^i + (\frac{1}{2}h_{ij}(u_0)\eta^i\eta^j)N(u_0)$$

- $h_{ij}\eta^i\eta^j$ positive definite $\Leftrightarrow k > 0 \Leftrightarrow$ elliptic paraboloid
- $h_{ij}\eta^i\eta^j$ indefinite $\Leftrightarrow k < 0 \Leftrightarrow$ hyperbolic paraboloid
- $h_{ij}\eta^i\eta^j$ semidefinite $\Leftrightarrow k = 0 \Leftrightarrow$ parabolic cylinder
- all $h_{ij} = 0 \Leftrightarrow k_1 = 0; k_2 = 0 \Leftrightarrow$ flat point (dt. Flachpunkt)



$K > 0$ $K = 0$ $K < 0$

Representation of the Surface

Another possibility to characterize local curvature is to use the so-called Dupin indicatrix, which in fact is strongly linked to the section curve. The section curve is obtained by intersecting the surface with planes that are close and parallel to the tangential plane. This section approximates a conic section that is homothetic to the Dupin indicatrix.

Representation of the Surface

Definition (Dupin Indicatrix)

By Euler's theorem $\mathfrak{X}(\lambda^1, \lambda^2) = k, \cos^2 \varphi + k_2 \sin^2 \varphi$ the following geometric constructs can be derived:

- Plot a straight-line segment of length $\frac{1}{|\mathfrak{X}|^{\frac{1}{2}}}$ from the origin of a planar Euclidean coordinate frame in each direction where the principal directions are used as coordinate axes $\rightarrow$

$$\pm 1 = k_1(z^1)^2 + k_2(z^2)^2$$

Representation of the Surface

Definition (Dupin Indicatrix) cont'd

- Put the indicatrix with its z^1- and z^2-axis respectively in the pricinipal directions of the corresponding tangential plane of the current surface point.
  1. Indicatrix is an ellipsis – elliptic point
  2. Indicatrix is pair of hyperbolae – hyperbolic point
  3. Indicatrix is pair of straight lines – parabolic point
  4. Indicatrix degenerates to a point – flat point

Representation of the Surface

Definition (Asymptotic Line)

The zero directions of the second fundamental form are called asymptotic directions. A surface curve with its tangent vectors pointing in the asymptotic direction at every point is called *asymptotic line*.

Asymptotic Line Remarks

1. The normal curvature of an asymptotic line vanishes ($k_n = 0$) if $k \neq 0$ the principal normal vector is orthogonal to the surface normal ($e_2 \perp N$).
2. Each straight line on a surface is an asymptotic line.
3. Asymptotic directions only exist for hyperbolic curvatures ($k < 0$).
4. Asymptotic directions are invariant under motions and parameter transformations.

Representation of the Surface

Definition (Curvature Line Parameters and Asymptotic Line Parameters)

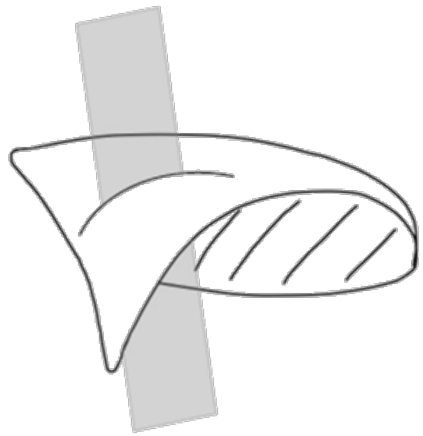
A parameterized surface $X(u, w)$ is called in relation to curvature line parameters or asymptotic line parameters respectively if the parameter lines $u^1 = const; u^2 = const$ are curvature lines or asymptotic lines respectively.

Curvature Line Parameters and Asymptotic Line Parameters Remarks

1. $X(u,w)$ is in relation to curvature line parameters iff $N_i = k_i X_i$.
   Equivalent to this is: $g_{12} = h_{12} = 0 \quad (h_{ii} = k_i g_{ii} \; i = 1,2)$.
   Curvature lines form a orthogonal network on all surfaces no matter how they are curved.
2. $X(u,w)$ is in relation to asymptotic line parameters iff $h_{11} = h_{22} = 0$.
   Asymptotic lines are not of high technical relevance. They only exist on surfaces with a negative curvature.

Curvature Line Parameters and Asymptotic Line Parameters Remarks cont'd

3. A surface curve is a curvature line iff the ruled surface (dt. Regelfläche) formed by the surface normal along the curve is a developable surface.



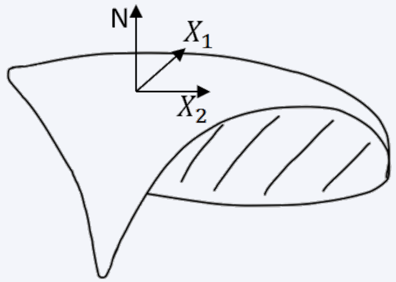
4. Curvature lines form an orthogonal curve grid on the surface.
5. Asymptotic lines form an orthogonal curve grid on a minimal surface.

Representation of the Surface

Trying to carry over the principle of the Frenet equations, we get the derivative equations and the integrability conditions:

Definition (Gaussian Frame)

Gaussian frame $\{X_1, X_2, N\}$



$$N_i = -\, h_i^k X_k$$
$$X_{ij} = \Gamma_{ij}^r X_r + h_{ij} N$$

with $\Gamma_{ij}^r = g^{rs}\Gamma_{ijs}$ and $\Gamma_{ijk} = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial u^i} + \frac{\partial g_{ik}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^k}\right)$

The derivative equations are the surface-theoretic analogon to the Frenet equations for curves.

Representation of the Surface

Definition (Integrability Conditions)

In curve theory, given continuous functions $k(s)$ (with $k > 0$) and $\tau(s)$ imply existence and uniqueness of a curve segment. The analogue problem is wether given $\{g_{ij}\}$ and $\{h_{ij}\}$ imply existence and uniqueness of a surface segment.

Representation of the Surface

Definition (Integrability Conditions) cond't

We need the so-called **integrability conditions**:

$$\frac{\partial X_{ij}}{\partial u^k} = \frac{\partial X_{ik}}{\partial u^j} \quad \text{and} \quad \frac{\partial N_i}{\partial u^j} = \frac{\partial N_j}{\partial u^i}$$

$$\frac{\partial \Gamma^r_{ij}}{\partial u^k} - \frac{\partial \Gamma^r_{ik}}{\partial u^j} + \Gamma^s_{kj}\Gamma^r_{sk} - \Gamma^s_{ik}\Gamma^k_{sj} = h_{ij}h^r_k + h_{ik}h^r_j$$

$$\frac{\partial h_{ij}}{\partial u^k} - \frac{\partial h_{ik}}{\partial u^j} + \Gamma^s_{ij}h_{sk} - \Gamma^s_{ik}h_{sj} = 0$$

(Christoffel-sysmbols are no geometric invariants like torsion and curvature!)

Representation of the Surface

Theorem (Fundamental Theorem of Surface Theory)

In a region $U \subset \mathbb{R}^2$ *let functions* $\tilde{g}_{ij}$ *of class* $\mathcal{C}^2$ *and functions* $\tilde{h}_{ij}$ *of class* $\mathcal{C}^1$ *be given with the following properties:*

1. $\tilde{g}_{ij} = \tilde{g}_{ji}$ *and* $\tilde{h}_{ij} = \tilde{h}_{ji}$
2. $(\lambda^1, \lambda^2) \mapsto \tilde{g}_{ij}\lambda^i\lambda^j$ *is positivie definite*
3. *the integrability conditions are satisfied*

then there is a parameterized surface $X : U \to \mathbb{E}^3$ *of class* $\mathcal{C}^3$ *with* $g_{ij} = \tilde{g}_{ij}$ *and* $h_{ij} = \tilde{h}_{ij}$.

Representation of the Surface

The situation of curvature can be defined with scalars like curvature, normal-section-curvature and mean curvature but it is also possible to define curvature tensors. A tensor is a generalized vector, invariant under parameter transformation.

Definition (Curvature Tensor)

$$R^r_{ikj} := \frac{\partial \Gamma^r_{ij}}{\partial u^k} - \frac{\partial \Gamma^r_{ik}}{\partial u^j} + \Gamma^s_{ij}\Gamma^r_{sk} - \Gamma^s_{ik}\Gamma^r_{sj}$$

Representation of the Surface

Theorema Egregium by Gauss

K only depends on the first fundamental form because one has: $K = \frac{R_{1212}}{g}$.

Introduction to Tensor Calculus

We will now briefly introduce some basics from tensor calculus to enable a better understanding of the curvature tensor and tensors in general.

Definition: Mathematical Model of a Tensor

Let V be an n-dim. vector space, V^* its dual space. A **tensor of type** (r, s) (r **contravariant** and s **covariant** indices) over V is a multilinear form T:

$$T : \underbrace{V \times V \times \ldots \times V}_{\text{r copies}} \times \underbrace{V^* \times V^* \times \ldots \times V^*}_{\text{s copies}} \to \mathbb{R}$$

$$r, s \in \mathbb{N}; \quad r + s > 0$$

$(r + s)$ is called the **order** of the tensor.

Introduction to Tensor Calculus

Applying such a multilinear map T of type (s, r) to a basis $\{E_1, \ldots, E_r\}$ for V and a canonical basis $\{E^1, \ldots, E^s\}$ for V^*, one obtains the following $(r+s)$-dim. array of components:

$$t^{j_1,\ldots,j_s}_{i_1,\ldots,i_r} := T(E_{i_1}, \ldots, E_{i_r}, E^{j_1}, \ldots, E^{j_s})$$

Such an array can be realized as the components of some multilinear map T. This motivates viewing multilinear maps as the intrinsic objects underlying tensors. There is also a component-free definition of the notion of a tensor. Using this, e.g. an intrinsic geometric statement (like the gaussian curvature) can be described by a tensor field on a manifold without the need to reference coordinates.

Introduction to Tensor Calculus

Note: Do not let yourself get confused by the following definitions speaking of type (s, r)-tensors instead of type (r, s). This is due to an error before translating the script into English, where r ans s have been swapped in the tensor definition. As the subscript always denotes the covariant and the superscript the contravariant indices, it seemed more convenient to prevent index errors in the formulas by swapping the variables in the type rather than the equations.

If there is time enough, the equations will be updated, though.

Introduction to Tensor Calculus

Applying such a multilinear map T of type (s, r) to a basis $\{E_1, \ldots, E_r\}$ for V and a canonical basis $\{E^1, \ldots, E^s\}$ for V^*, one obtains the following $(r+s)$-dim. array of components:

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Introduction to Tensor Calculus

Remarks

1. The elements of V^* are tensors of type $(0, 1)$. They are called **covariant vectors**.
2. The elements of V can be identified with the tensors of type $(1, 0)$ as V^{**} and V are canonically isomorphic. They are called **contravariant vectors**.
3. The set $T(r, s)$ of type (r, s) tensors consitutes a vector space with "the usual relations" of dimension $r + s$.
4. Because of their special behavior under transformations, tensors are a convenient tool for the description of "geometrically technical" properties.

Introduction to Tensor Calculus

notations: tensor components $\leftrightarrow$ "coordinates"

Let $\{E_1, \dots, E_n\}$ denote a basis of V and $\{E^1, \dots, E^n\}$ the dual basis of V^*, i.e.

$E^i E_j = \delta^i_j$; (Kronecker delta: $\delta = 1$ if $i = j$, else $\delta = 0$)

$A_\nu = a^j_\nu E_j$ (contravariant: "upper index j"; $\in V$);

$B^\nu = b^\nu_j E^j$ (covariant: "lower index j"; $\in V^*$)

$$T(A_1, \dots A_r, B^1, \dots, B^s) = T(a^{i_1}_1 E_{i_1}, \dots, a^{i_r}_r E_{i_r}, b^1_{j_1} E^{j_1}, \dots, b^s_{j_s} E^{j_s})$$
$$= T(E_{i_1}, \dots, E_{i_r}, E^{j_1}, \dots, E^{j_s}) a^{i_1}_1 \cdots a^{i_r}_r b^1_{j_s} \cdots b^s_{j_s}$$
$$= t^{j_1, \dots, j_s}_{i_1, \dots, i_r} a^{i_1}_1 \cdots a^{i_r}_r b^1_{j_s} \cdots b^s_{j_s}$$
$$= \tilde{t}^{k_1, \dots k_s}_{l_1, \dots l_r} \tilde{a}^{l_1}_{(1)} \cdots \tilde{a}^{l_r}_{(r)} \tilde{b}^{(1)}_{k_1} \cdots \tilde{b}^{(s)}_{k_s}$$

The n^{r+s} numbers $t^{j_1, \dots, j_s}_{i_1, \dots, i_r}$ are called the **coordinates** or **components** of the tensor T.

Introduction to Tensor Calculus

Transformation Behavior of Tensor Components under Basis Transformation:

Let $\tilde{E}_i = \alpha_i^j E_j$, resp. $\tilde{E}^i = \breve{\alpha}_j^i E^j$ with $\{\breve{\alpha}_j^i\}$ inverse to $\{\alpha_i^j\}$

$\tilde{a}^i = a^r \breve{\alpha}_r^i$ "contravariant"; $\tilde{b}_i = b_r \alpha_i^r$ "covariant"

The tensor components transform like the basis!

$$A_{(1)}, \dots, A_{(r)} \in V; \quad B^{(1)}, \dots, B^{(s)} \in V^*$$

$$T(A_{(1)}, \dots, A_{(r)}, B^{(1)}, \dots, B^{(s)}) = t_{i_1,\dots,i_r}^{j_1,\dots,j_s} a_{(1)}^{i_1} \cdots a_{(r)}^{i_r} b_{j_1}^{(1)} \cdots b_{j_s}^{(s)}$$

$$= \tilde{t}_{l_1,\dots l_r}^{k_1,\dots k_s} \tilde{a}_{(1)}^{l_1} \cdots \tilde{a}_{(r)}^{l_r} \tilde{b}_{k_1}^{(1)} \cdots \tilde{b}_{k_s}^{(s)}$$

$$= \tilde{t}_{l_1,\dots l_r}^{k_1,\dots k_s} \breve{\alpha}_{i_1}^{l_1} \cdots \breve{\alpha}_{i_r}^{l_r} \alpha_{k_1}^{j_1} \alpha_{k_s}^{j_s} a_{(1)}^{i_1} \cdots a_{(r)}^{i_r} b_{j_1}^{(1)} \cdots b_{j_s}^{(s)}$$

therefore, we get the following transformation behavior:

Introduction to Tensor Calculus

Transformation Behavior of Tensor Components under Basis Transformation:

$$\tilde{t}^{i_1,\dots i_l}_{j_1,\dots j_s} = t^{k_1,\dots,k_s}_{l_1,\dots,l_r} \alpha^{l_1}_{i_1} \cdots \alpha^{l_r}_{i_r} \check{\alpha}^{j_1}_{k_1} \cdots \check{\alpha}^{j_s}_{k_s}$$

The arc length $\left(\frac{ds}{dt}\right)^2 = g_{ij}\frac{du^i}{dt}\frac{du^j}{dt}$ satisfies this situation.

Theorem: Tensor Transformation Behavior

Let there be n^{r+s} numbers $t^{j_1,\dots,j_s}_{i_1,\dots,i_r}$ for every base of V. These systems constitue the components of a tensor of type (r,s) iff. the transformation behavior defined above holds.

Introduction to Tensor Calculus

Tensor Operations

- Tensors of same type can be added and be multiplied with scalars.
- Tensor product: Let T (s,r)-tensor; $\tilde{T}$ $(\tilde{s},\tilde{r})$-tensor

$$T\tilde{T}(A_1,\ldots,A_{r+\tilde{r}},B^1,\ldots,B^{s+\tilde{s}})$$
$$:=T(A_1,\ldots,A_r,B^1,\ldots B^s)\cdot\tilde{T}(A_{r+1}\ldots A_{r+\tilde{r}},B^{s+1},\ldots B^{s+\tilde{s}})$$

is called the **tensor product of T and $\tilde{T}$**. In components:

$$(t\tilde{t})^{j_1,\ldots,j^{s+\tilde{s}}}_{i_1,\ldots,i_{r+\tilde{r}}}:=t^{j_1,\ldots,j_s}_{i_1,\ldots,i_r}\tilde{t}^{j_{s+1},\ldots,j_{s+\tilde{s}}}_{i_{r+1},\ldots,i_{r+\tilde{r}}}$$

$T\tilde{T}$ is a tensor of type $(s+\tilde{s},r+\tilde{r})$.

Introduction to Tensor Calculus

Tensor Operations

- Contraction: Let T (s, r)-tensor, where $r, s, \geq 1$.

$$t^{j_1,\dots,j_s}_{i_1,\dots,i_r} \mapsto t^{j_l}_{i_k} \cdot t^{j_1,\dots,j_{l-1},j_{l+1},\dots,k_s}_{i_1,\dots,i_{k-1},i_{k+1},\dots,i_r}$$

  is called the **tensor contraction** of T with respect to the k-th covariant and the l-th contravariant index. This procedure yields a tensor of type $(s-1, r-1)$.
  Example: $g^{ij}; h_{rs} \xrightarrow[mult]{} g^{ij} h_{rs} \mapsto g^{ij} h_{is} \mapsto g^{ij} h_{ij} = 2h$
- **raising** and **lowering** indices with respect to a symmetric, non-degenerate fundamental (metric) tensor:

$$t^{j_1,\dots,j_s}_{i_1,\dots,i_r} \mapsto g^{kl} t^{j_1,\dots j_s}_{l,i_2,\dots,i_r} =: t^{k,j_1,\dots,j_s}_{i_1,\dots,i_r}$$
$$t^{j_1,\dots,j_s}_{i_1,\dots,i_r} \mapsto g_{kl} t^{l,j_2,\dots,j_s}_{i_1,\dots,i_r} =: t^{j_2,\dots,j_s}_{i_1,\dots,i_r,k}$$

Introduction to Tensor Calculus

Example for raising an index:
One gets the matrix of the Weingarten map by raising an index in the matrix of the second fundamental form with g^{rj}:

$$h_i^j = h_{ir} g^{rj}$$

Introduction to Tensor Calculus

Covariant Differentiation and Parallelity:

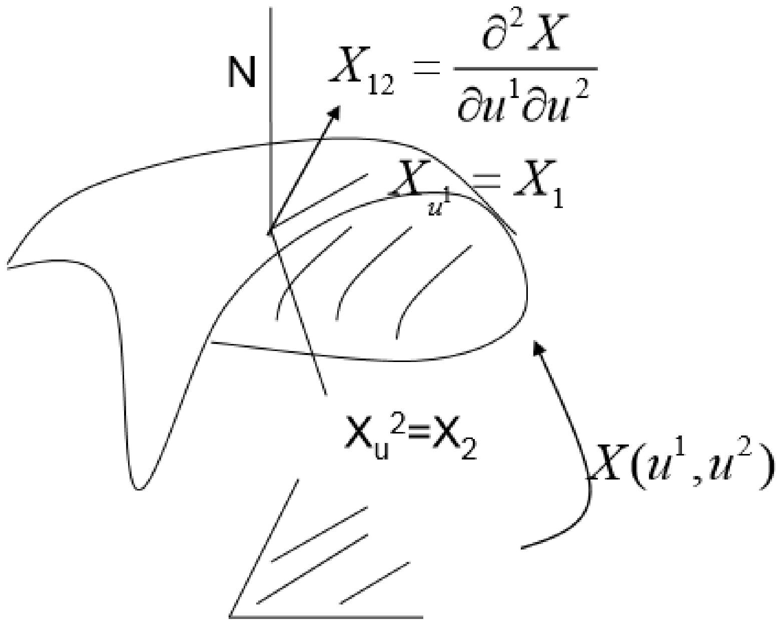
The usual partial differentiation of a tensor does not yield another tensor. One thus has to come up with a more general method. Such a method is given by the **covariant differentiation** which we will discuss in the following.
The process of differentiation is to be formulated in a way that makes an arbitrary derivative of a tensor yield another tensor. This means nothing else than that this differentiation has to be independent from the coordinates.

Introduction to Tensor Calculus

Motivation:

$X_{ij} = \Gamma^r_{ij} X_r + h_{ij} N,$

where $\Gamma_{ijk} = \langle X_{ik}, X_j \rangle = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial u^i} + \frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{ij}}{\partial u^k}\right)$ and $\Gamma^r_{ij} = g^{rs}\Gamma_{ijs}$



Introduction to Tensor Calculus

Motivation:

The Christoffel symbols Γ_{ij}^{k} are **not** components of the tensor!

$$\Gamma_{ij}^{k} = \tilde{\Gamma}_{rs}^{l} \frac{\partial \tilde{u}^r}{\partial u^i} \frac{\partial \tilde{u}^s}{\partial u^j} \frac{\partial u^k}{\partial \tilde{u}^l} + \frac{\partial^2 \tilde{u}^l}{\partial u^i \partial u^j} \frac{\partial u^k}{\partial \tilde{u}^l}$$

but $X_{ij} - \Gamma_{ij}^{r} X_r = h_{ij} N$ is geometrically invariant.
In the context of the inner geometry of a surface, only vector fields Z that are tangential along the plane are of interest. The derivatives will in general not be tangential vector fields anymore. Therefore, we define:

$$\frac{\nabla Z}{dt} := \frac{dZ}{dt} - \left\langle \frac{dZ}{dt}, N \right\rangle N$$

With these motivations, we now turn to the exact definitions:

Introduction to Tensor Calculus

Theorem and Definition: Covariant Derivative of a Tensor

Let A an (s,r)-tensor with components $a^{q_1,\dots,q_s}_{l_1,\dots,l_r}$. The numbers

$$a^{q_1,\dots,q_s}_{l_1,\dots,l_r}\Big|_i = \frac{\partial a^{q_1,\dots,q_s}_{l_1,\dots,l_r}}{\partial u^i}$$

$$-\sum_{m=1}^{r} a^{q_1,\dots,q_s}_{l_1,\dots,l_{m-1},k,l_{m+1},\dots,l_r}\Gamma^k_{il_m}$$

$$+\sum_{m_1}^{s} a^{q_1,\dots,q_{m-1},p,q_{m+1},\dots,q_s}_{l_1,\dots,l_r}\Gamma^{q_m}_{pi}$$

are the components of an $(s,r+1)$-tensor. This tensor is called the **covariant derivative** of the tensor $a^{q_1,\dots,q_s}_{l_1,\dots,l_r}$ with respect to g_{ik}.

Introduction to Tensor Calculus

Important Special Cases:

- $a^k_j\big|_i = \frac{\partial a^k_j}{\partial u^i} - a^k_r \Gamma^r_{ij} + a^s_j \Gamma^k_{si}$
- $a_{jk}\big|_i = \frac{\partial a_{jk}}{\partial u^i} - a_{rk} \Gamma^r_{ij} + a_{js} \Gamma^s_{ki}$
- $a^{jk}\big|_i = \frac{\partial a^{jk}}{\partial u^i} - a^{rk} \Gamma^j_{ri} + a^{js} \Gamma^k_{si}$
- Gaussian derivative equations: $X\big|_{ij} = h_{ij} N$

A very important application of the covariant differentiation is the definition and determination of parallels and geodesics.

Parallelity and Levi-Civita Connection

Definition: Covariant Derivative of a Vector Field

Let $X_U \to \mathbb{R}^3$ a surface, $C = X \circ U : I \to \mathbb{E}^3$ a surface curve, and $Z : I \to \mathbb{E}^3$ a tangential vector field along C, i.e. $Z(t) \in T_{c(t)}X$.
The **covariant derivative of** Z **in** t is the vector

$$p_r \circ \frac{dZ(t)}{dt} =: \frac{\nabla dZ(t)}{dt}$$

where $p_{r(t)} : T_{c(t)}\mathbb{R}^3 \to T_{u(t)}X$.

Parallelity and Levi-Civita Connection

Theorem:

The covariant derivative $\frac{dZ(t)}{dt}$ of a tangential vector field $Z(t)$ along the surface curve $C(t) = X \circ U(t)$ is independent from the choice of parameters for the surface. This derivative is an object of the inner surface geometry.
For $Z(t) = \xi^k(t) X_k \circ U(t)$, one obtains:

$$\frac{dZ(t)}{dt} = \dot{\xi}^k X_k + \xi^i \dot{u}^j (\Gamma_{ij}^k X_k + h_{ij} N) \quad \text{and thus:}$$

$$\frac{\nabla Z(t)}{dt} \equiv \{\dot{\xi}^k(t) + \xi^i(t) \dot{u}^j(t) \Gamma_{ij}^k \circ U(t)\} X_k \cdot U(t)$$

Parallelity and Levi-Civita Connection

Directed Covariant Derivative

The terminology defined above can be used to define the covariant derivative of the tangential vector field Z in the direction of a tangential vector field Y:

$$\nabla Z = T_U X \to T_U X$$
$$y \mapsto \frac{\nabla Z \circ (U(0))}{dt}$$

where $\frac{\nabla Z \circ (U(0))}{dt} = \left(\frac{\partial \xi^k}{\partial u^j}(u(0)) + \xi^i(u(0))\Gamma^k_{ij}(u(0))\right)\dot{u}^j(0)X_k$,
$y = \eta^j X_j$, and $c(t) = X \circ u(t)$ is a surface curve with
$u^j(0) = u^j_0$; $\dot{u}^j(0) = \eta^j$, and $z = \xi^k X_k$; especially, we have
$\frac{\nabla Z(t)}{dt} = \nabla Z\left(\dot{c}(t)\right)$

Parallelity and Levi-Civita Connection

Definition: Parallelity

Let $c(t) = X \circ u(t)$ a surface curve and $Z(t)$ a vector field along c. Z is called **parallel along** c if $\frac{dZ(t)}{dt} = 0$.

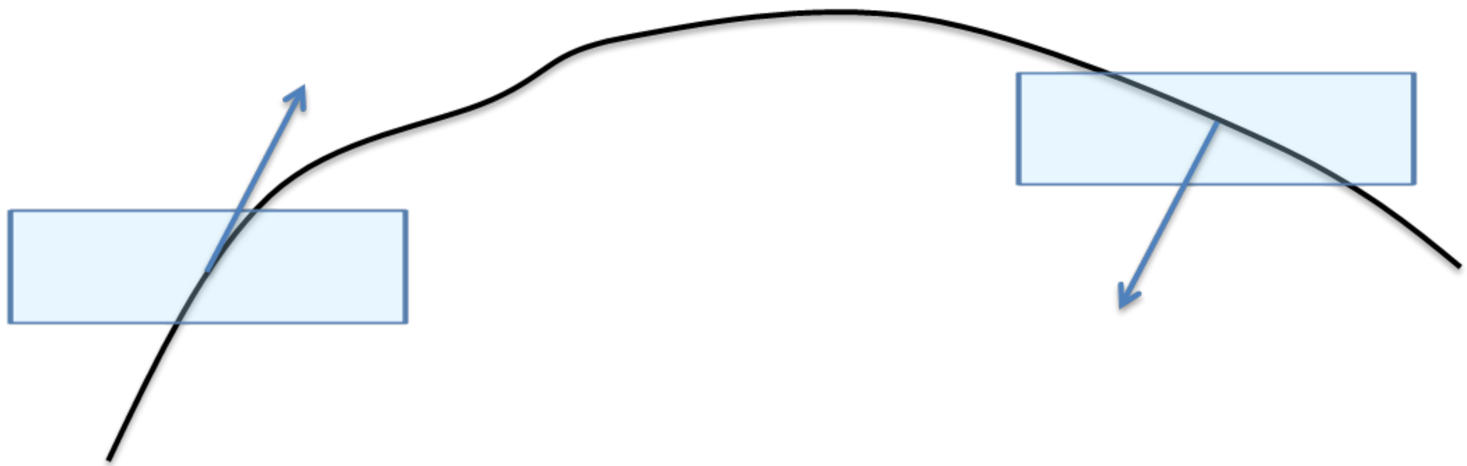
Differential Equation of Parallel Transport:
$Z(t) = \xi^i(t) X_i \circ u(t)$ is parallel along $c(t)$ iff.
$\dot{\xi}^k(t) + \xi^i(t)\dot{u}^j(t)\Gamma_{ij}^k \circ u(t) = 0.$

This definition is a generalization of the notion of parallelity in the plane, as if $X(t)$ and $Y(t)$ are parallel vector fields along c, $g_{c(t)}(x, y) = const.$, i.e. the angles are equal for all values of the parameter t.

Parallelity and Levi-Civita Connection

The Levi-Civita-Connection:
The differential equation for parallel transport defines some kind of "connection" that enables the transport of given tangential vectors ($A = a^j x_j$) along a surface curve. This connection is called the **Levi-Civita connection** and the equations for parallel transport are called the **equations for the Levi-Civita connection**.

Parallelity and Levi-Civita Connection



The Levi-Civita connection arises from the aim to compare tangent vectors bound to different points of a surface with respect to their direction. The connection is a map that allows to transport tangent vectors from one point of a surface S along a curve C on S to another point on F. The vectors obtained this way are called parallel along C with respect to the connection.

Parallelity and Levi-Civita Connection

Properties of the Levi-Civita Connection:

- The length of a surface vector does not change under the Levi-Civita connection.
- The angle enclosed by two surface vectors bound in the same point does not chance if both vectors are transported along the same curve.
- **The Levi-Civita connection is "compatible with the metric"**

Parallelity and Levi-Civita Connection

Under which conditions is the parallel transport independent from the path?

Theorem: Path Independence of Parallel Transport

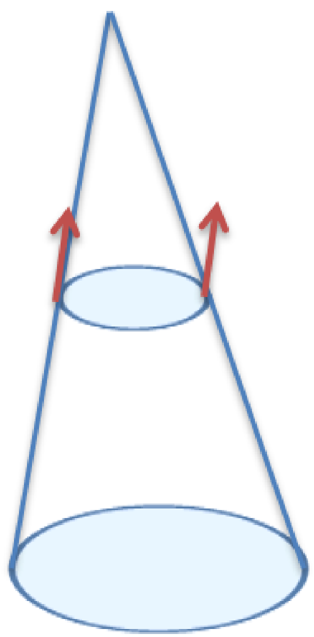
Let $X : U \to \mathbb{E}^3$ be a surface. Then, the following statements are equivalent:

- The Gaussian curvature vanishes, i.e. $K(u) = 0$
- One can introduce local parameters with $g_{ij} = \delta_{ij}$
- The parallel transport is path-independent
- $X(u)$ is locally isometric to an open subset of the Euclidian plane $\mathbb{E}^2$

Parallelity and Levi-Civita Connection

Remarks:

- These statements do *not* hold globally!
- If two surfaces X and $\tilde{X}$ have the same constant curvature, they are locally isometric:

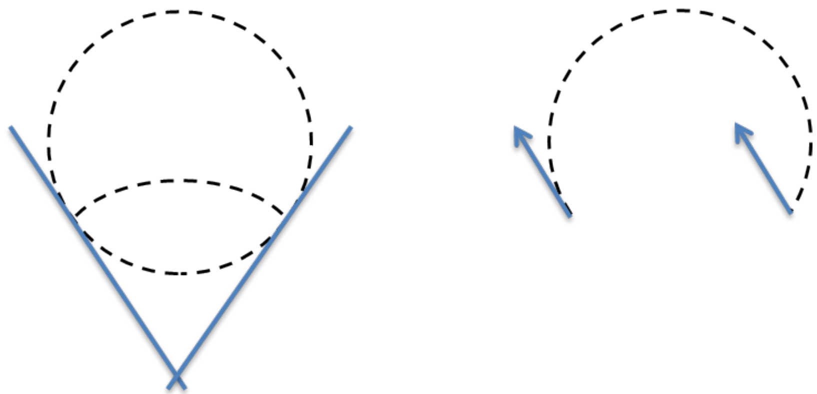


Parallelity and Levi-Civita Connection

Based on these results, we give another illustration of the Levi-Civita connection:
The Levi-Civita connection is an inner property of the surface. However, one can of course include the embedding of the surface in space into the consideration. Due to the invariance of the connection for length-preserving maps, one can perform transport of a vector along a curve C on a surface S lying in Euclidian space in a geometrically descriptive manner:

Parallelity and Levi-Civita Connection

Choose a developable surface (= torse) T (i.e. a surface with vanishing Gaussian curvature $K = 0$; either a cylinder, a cone or a tangent plane), that touches the surface along the surface curve C. Map T to the plane preserving the lengths and move the given surface vector parallel along the image C^* of C. Now, map the vectors obtained this way back to T by the inverse of the length-preserving map used to map T to the plane. These vectors are now identical to the surface vectors of S obtained by the transport of a given surface vector of S along the surface curve C.



Geodesics, Developable and Minimal Surfaces

We now focus on the **geodesic curvature** kg:

For a surface curve $y(t) = X(u^1(t), u^2(t))$, we have $y'' = k_n \cdot N + kg[N, y']$ and thus $kg = \det(N, y', y'') = |N, y', y''|$.

The absolute value of the geodesic curvature $|kg|$ is equal to the curvature k^* along curve $y^*(t)$ (in point P), where y^* is the orthogonal projection of $y(t)$ into the tangential plane (in point P).

The geodesic curvature is determined by the first fundamental form and therefore a varibale of the inner geometry.

Geodesics, Developable and Minimal Surfaces

Definition: Geodesic Lines

A curve that lies inside a surface and whose geodesic curvature kg vanishes identically, is called a **geodesic line**.

$$kg = 0 \rightarrow y'' = k_n N \rightarrow \frac{\nabla y'(t)}{dt} = 0$$

Differential equation of geodesic lines:

$$\ddot{u}^k - \Gamma^k_{ij} \dot{u}^i \dot{u}^j = 0$$

One directly finds that **geodesics are autoparallels** (i.e. $\xi^i = \dot{u}^i$).

Geodesics, Developable and Minimal Surfaces

Applying variational calculus to answer the question for the shortest part of a curve that connects two points yields the corresponding Cauchy-Euler euqation (sometimes called Euler's equation or Euler's differential equation):

$$-\frac{\partial h}{\partial f^i} + \frac{d}{dt}\left(\frac{\partial h}{\partial \dot{f}^i}\right) = 0, \quad \text{where } f^i = u^i \text{ and } h = \sqrt{g_{ij}\dot{f}^i\dot{f}^j}$$

One directly finds that this again yields the differantial equation for geodesic lines.

The **shortest connection** of two points on a surface is **part of a geodesic line**.

In every point, in every direction exactly one geodesic starts!

Geodesics, Developable and Minimal Surfaces

Geodesics are not only lines with vanishing geodesic curvatue and autoparallels but also locally shortest connections of points!

Utilizing geodesics, one can introduce special coordinate systems analoguous to planar orthogonal systems or planar polar coordinates. Such coordinate systems serve well for different application purposes.

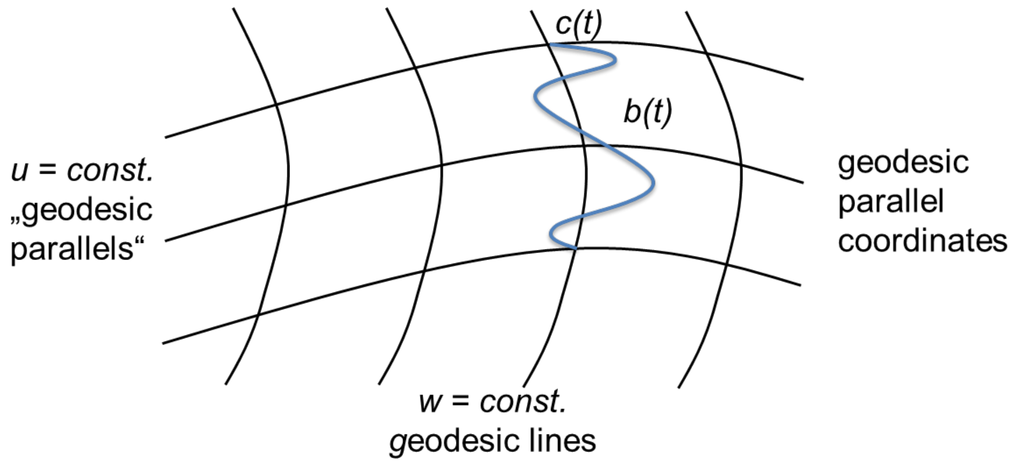


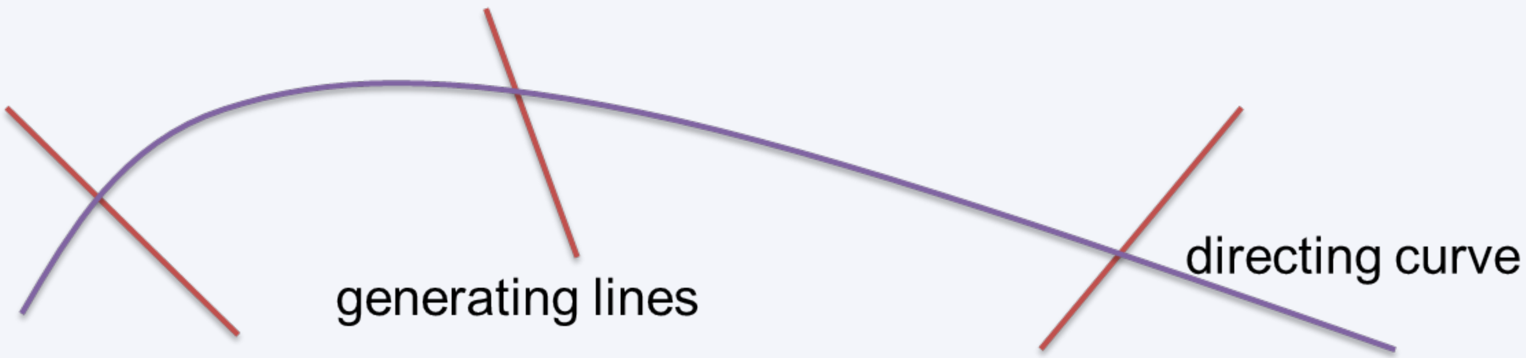
Figure: Geodesic coordinates. $L(b) \geq L(c)$ if $b(t)$ lies completely inside the "geodesic field".

Geodesics, Developable and Minimal Surfaces

Special Surfaces

Ruled Surface, Developable Surface

A surface is called a **ruled surface** if it can be locally parameterized in the form $X(u,w) = y(u) + w \cdot z(u)$.



A ruled surface is called a **developable surface** (sometimes: torse) if the normal vector is constant along each generating line. This is the same as zero Gaussian curvature.

Geodesics, Developable and Minimal Surfaces

Theorem: Classification of Developable Surfaces

- A ruled surface is developable iff. for every natural parameterization (= arc length parameterization) one has $X(u,w) = y(u) + w \cdot z(u)$.
- $\det(x_1, x_2, x_{12}) = 0$.
- A $\mathcal{C}^3$-surface without flat points (i.e. without points where the torsion vanishes) is developable iff. the Gaussian curvature vanishes ($K \equiv 0$).
- A developable surface is either a cylinder, a cone, or a tangent plane.

Geodesics, Developable and Minimal Surfaces

Minimal Surfaces:

The necessary condition for a surface $X(u)$ to have minimal area among the surfaces to be compared is $H \equiv 0$, where H denotes the mean curvature. A surface with vanishing mean curvature is called a **minimal surface**.

Remarks

1) Surface parts of minimal area that have a fixed boundary are always parts of minimal surfaces.
2) For every simply closed Jordan curve there exists at least one minimal surface part whose boundary is said curve.
3) A surface of class $r \geq 3$ is a minimal surface or the surface of a sphere iff. its spherical map is a conformal map (i.e. it preserves angles locally).

Geodesics, Developable and Minimal Surfaces

Remarks

4) Minimal surfaces cannot possess elliptical points.
5) Let $X(u)$ be based on isothermic parameters ($g_{11} = g_{22}$; $g_{12} = 0$). Then, $X(u)$ is a minimal surface iff $X_{11} + X_{22} = 0$. The solutions to this Laplace differential equation are harmonic functions and thus the real part of a holomorphic function and thereby a real-valued analytic function. Therefore, minimal surfaces are analytic surfaces!
6) The possibility to introduce isothermic parameters on a surface M with Riemannian metric implies that M can be provided with a complex structure that makes M a Riemannian surface with respect to complex analysis (the theory of functions of a complex variable).