Geometric Modelling Summer 2018

Prof. Dr. Hans Hagen

http://hci.uni-kl.de/teaching/geometric-modelling-ss2018

Variational Design

Polynomial Bézier Curves

"Energy Smoothing" has been extended to a methodology of variational design. In this chapter, we first discuss polynomial Bézier curves:

$$X_l(u) := \sum_{i=0}^{m} b_{l+i} B_i^m \left(\frac{u - u_l}{u_{l+1} - u_l} \right)$$

Polynomial Bézier Curves

The design algorithm combines least squares approximation with an automated smoothing process. The basic mathematical model is the following variation principle:

$$
\begin{aligned}
&(1 - w_s) \cdot \sum_{j=1}^{n_p} w_{pj} \left(X(t_{pj}) - P_j\right)^2 \\
&+ w_s \left[w_{2,g} \left(\sum_{i=1}^{n} w_{2,i} \int_{t_i}^{t_{i+1}} \|X''(t)\|^2 dt \right) \right. \\
&\left. + (1 - w_{2,g}) \left(\sum_{i=1}^{n} w_{3,i} \int_{t_i}^{t_{i+1}} \|X'''(t)\|^2 dt \right) \right]
\end{aligned}
$$

For a curve X to me modelled using poinjts P and weights w.

Polynomial Bézier Curves

$X(t)$ is the parametric representation of the curve, $t \in [t_1, t_{n+1}]$, and n is the number of curve segments.

$\{P_j\}_{j=1}^{np}$ points to be approximated, $w_s, w_{2,g}, w_{2,i}, w_{3,i} \in [0, 1]$, where $\sum_{i=1}^{n} w_{2,i} = 1$ and $\sum_{i=1}^{n} w_{3,i} = 1$.

Polynomial Bézier Curves

This variational principle is now applied to quintic $\mathcal{C}^2$ Bézier curves:

$$X_i(t) = \sum_{k=0}^{5} b_k^i B_k^5 \left(\frac{t - t_i}{t_{i+1} - t_i}\right)$$

where the following matching conditions have to be met:

- $\mathcal{C}^0$-cont.: $b_5^i = b_0^{i+1}$
- $\mathcal{C}^1$-cont.: $\frac{5}{\Delta t_i}(b_5^i - b_4^i) = \frac{5}{\Delta t_{i+1}}(b_i^{i+1} - b_2^{i+1})$
- $\mathcal{C}^2$-cont.: $\frac{20}{\Delta^2 t_i}(b_3^i - 2b_4^i + b_5^i) = (b_0^{i+1} - 2)$

The control points b_i^k are used as input parameters for the variational principle.

Polynomial Bézier Curves

The variational process consists of three steps:

1) **Least Squares Fitting**:

$$SL_i := \sum_{i=1}^{n} \sum_{j \in I_i} w_{pj} (X_i(t_{pj}) - P_j)^2 \to \min$$

$$I_i := \{j | t_{pj} \in [t_i, t_{i+1}]\}$$

In Bézier representation, this is the following:

$$\sum_{i=1}^{n} \sum_{j \in I_i} w_{pj} \left(\sum_{k=0}^{5} b_k^i B_k^5(t_{pj}) - P_j \right)^2 \to \min$$

Polynomial Bézier Curves

The neccessary condition $\frac{\partial SL}{\partial b_k^i} = 0,\ 0 \leq k \leq 5,\ 1 \leq i \leq n$ yields the linear equation system:

$$\frac{\partial SL}{\partial b_l^i} = \sum_{k=0}^{5} b_k^i S_{lk}^i - D_l^i = 0$$

where $S_{lk}^i := 2 \sum_{j \in I_i} w_{pj} B_k^5(t_{pj}) B_l^5(t_{pj})$,
$D_l^i := 2 \sum_{j \in I_i} w_{pj} P_j B_l^5(t_{pj})$, and $i \in \{1, \ldots, n\}$, and
$l, k \in \{0, \ldots, 5\}$.
The matching conditions are yet to be implemented.

Polynomial Bézier Curves

2) Automated Smoothing Process

Smoothing criterion: $w_{2,g} I_2 + (1 - w_{2,g}) I_3 \to \min$

$$I_2 := \sum_{i=1}^{n} w_{2,i} \int_{t_i}^{t_{i+1}} \|X''(t)\|^2 dt$$

$$I_3 := \sum_{i=1}^{n} w_{3,i} \int_{t_i}^{t_{i+1}} \|X'''(t)\|^2 dt$$

Polynomial Bézier Curves

The variation steps yield the following equation systems:
For l_2, for every segment, one gets:

$$
\begin{aligned}
\frac{\partial l_2^i}{\partial b_0^i} &= \frac{400}{7}\cdot\frac{1}{(\Delta t_i)^3}\cdot\left(2b_0^i - 3b_1^i + \frac{2}{5}b_2^i + \frac{3}{10}b_3^i + \frac{1}{5}b_4^i + \frac{1}{10}b_5^i\right) = 0\\
\frac{\partial l_2^i}{\partial b_1^i} &= \frac{400}{7}\cdot\frac{1}{(\Delta t_i)^3}\cdot\left(-3b_0^i + \frac{26}{5}b_1^i - \frac{13}{10}b_2^i - \frac{4}{5}b_3^i - \frac{3}{10}b_4^i + \frac{1}{5}b_5^i\right) = 0\\
\frac{\partial l_2^i}{\partial b_2^i} &= \frac{400}{7}\cdot\frac{1}{(\Delta t_i)^3}\cdot\left(\frac{2}{5}b_0^i - \frac{13}{10}b_1^i + \frac{6}{5}b_2^i + \frac{1}{5}b_3^i + \frac{4}{5}b_4^i + \frac{3}{10}b_5^i\right) = 0\\
\frac{\partial l_2^i}{\partial b_3^i} &= \frac{400}{7}\cdot\frac{1}{(\Delta t_i)^3}\cdot\left(\frac{3}{10}b_0^i - \frac{4}{5}b_1^i + \frac{1}{5}b_2^i - \frac{6}{5}b_3^i - \frac{13}{10}b_4^i + \frac{2}{5}b_5^i\right) = 0\\
\frac{\partial l_2^i}{\partial b_4^i} &= \frac{400}{7}\cdot\frac{1}{(\Delta t_i)^3}\cdot\left(\frac{1}{5}b_0^i - \frac{3}{10}b_1^i - \frac{4}{5}b_2^i - \frac{13}{10}b_3^i + \frac{26}{5}b_4^i - 3b_5^i\right) = 0\\
\frac{\partial l_2^i}{\partial b_5^i} &= \frac{400}{7}\cdot\frac{1}{(\Delta t_i)^3}\cdot\left(\frac{1}{10}b_0^i + \frac{1}{5}b_1^i + \frac{3}{10}b_2^i + \frac{2}{5}b_3^i - 3b_4^i + 2b_5^i\right) = 0
\end{aligned}
\tag{1}
$$

Polynomial Bézier Curves

... and for I_3, one obtains:

$$
\begin{aligned}
\frac{\partial I_3'}{\partial b_0'} &= \frac{720}{(\Delta t_i)^3} \cdot \left(2b_0^i - 5b_1^i + \frac{10}{3} b_2^i - \frac{1}{3} b_5^i \right) = 0 \\
\frac{\partial I_3'}{\partial b_1'} &= \frac{720}{(\Delta t_i)^3} \cdot \left(-5b_0^i + \frac{40}{3} b_1^i - 10b_2^i + \frac{5}{3} b_4^i \right) = 0 \\
\frac{\partial I_3'}{\partial b_2'} &= \frac{720}{(\Delta t_i)^3} \cdot \left(\frac{10}{3} b_0^i - 10b_1^i + 10b_2^i - \frac{10}{3} b_3^i \right) = 0 \\
\frac{\partial I_3'}{\partial b_3'} &= \frac{720}{(\Delta t_i)^3} \cdot \left(-\frac{10}{3} b_2^i + 10b_3^i - 10b_4^i + \frac{10}{3} b_5^i \right) = 0 \\
\frac{\partial I_3'}{\partial b_4'} &= \frac{720}{(\Delta t_i)^3} \cdot \left(\frac{5}{3} b_1^i - 10b_3^i + \frac{40}{3} b_4^i - 5b_5^i \right) = 0 \\
\frac{\partial I_3'}{\partial b_5'} &= \frac{720}{(\Delta t_i)^3} \cdot \left(-\frac{1}{3} b_0^i + \frac{10}{3} b_3^i - 5b_4^i + 2b_5^i \right) = 0
\end{aligned} \tag{2}
$$

Like in the least suqares step, the matching conditions are yet to be implemented.

Polynomial Bézier Curves

3) Merging

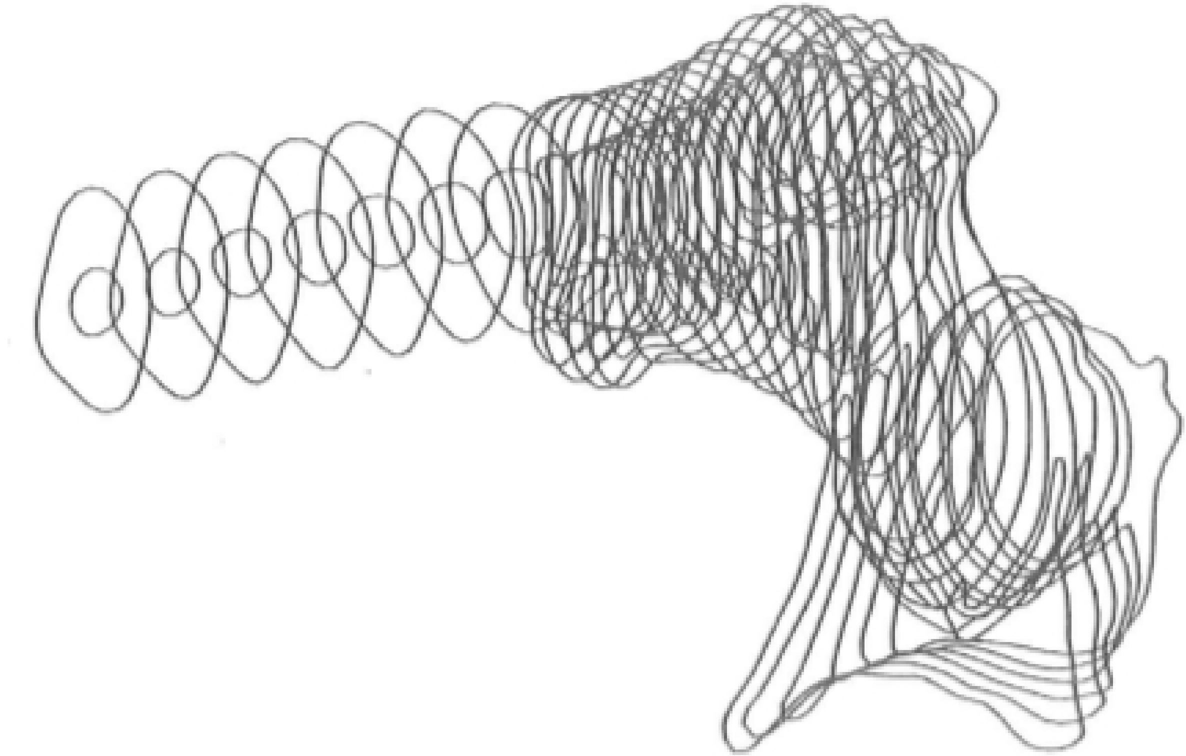
Now, we combine the steps to an overall concept:

$$(1 - w_s)A + w_s B = 0$$

where A stands for the equations of the linear equation system obtained from least squares fitting and B represents the equation for l_2 and l_3 from the smoothing step.

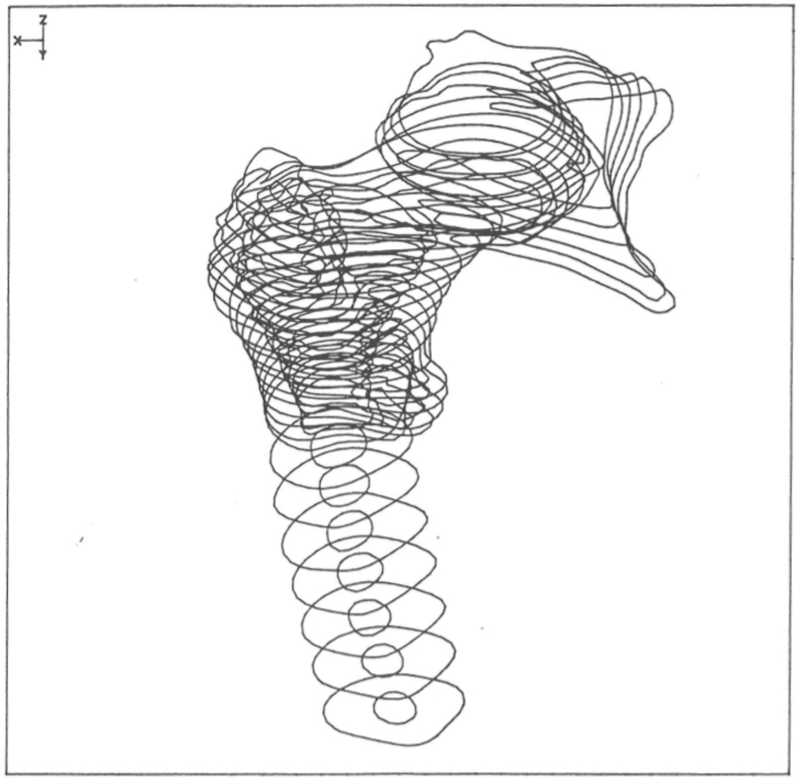
Polynomial Bézier Curves

Application: Contour Curves for Medical Imaging Data:



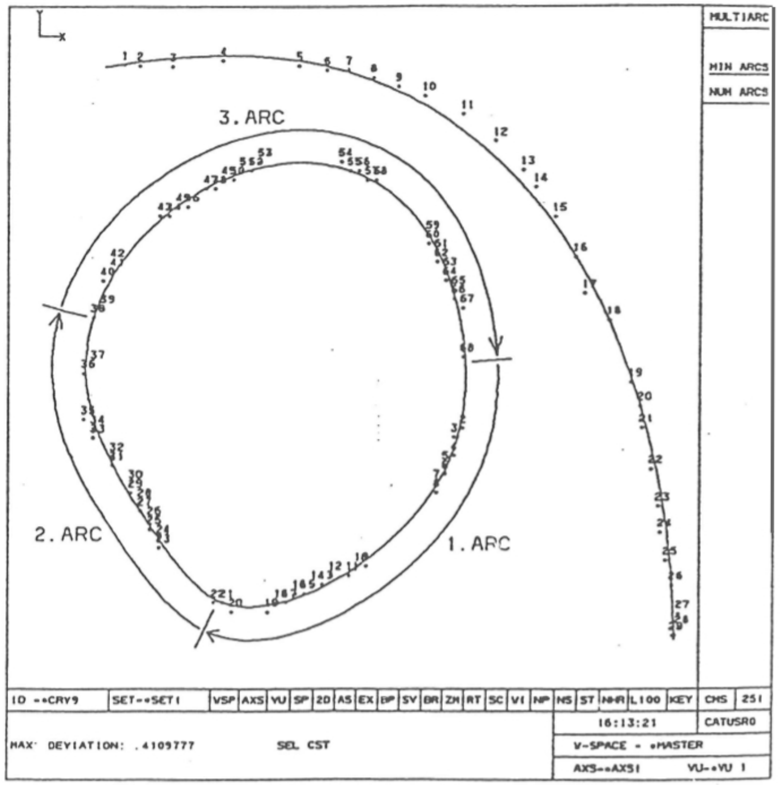
Polynomial Bézier Curves

Application: Contour Curves for Medical Imaging Data:



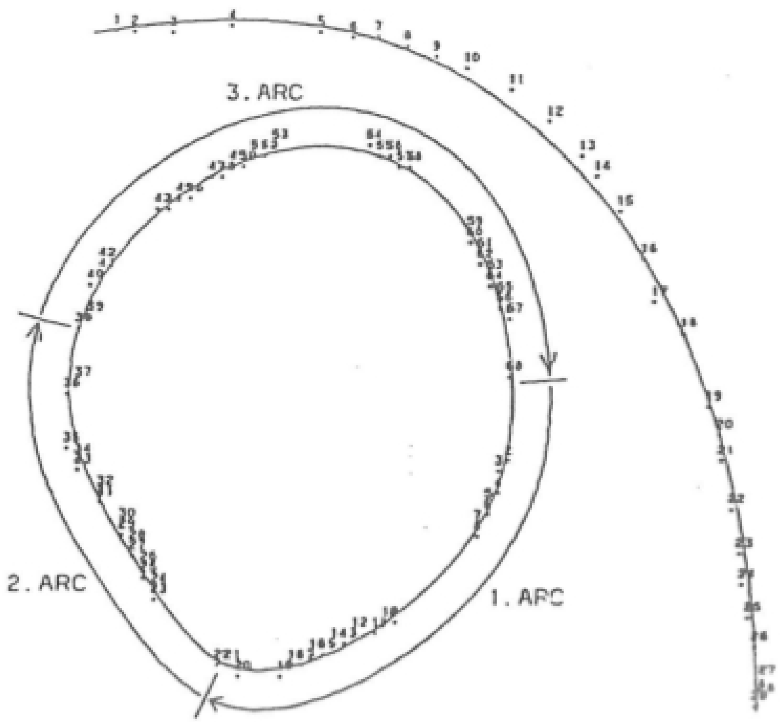
Polynomial Bézier Curves

Application: Contour Curves for Medical Imaging Data:



Polynomial Bézier Curves

Application: Contour Curves for Medical Imaging Data:
The algorithm works for closed as well as open curves:



B-Spline Curves

Here, the have the following mathematical model as a basis:

$$(1 - w_s) \sum_{j=1}^{n_p} w_{pj}(X(t_{pj}) - P_j)^2 + w_s \sum_{i=1}^{n} w_{3,i} \sum_{t_i}^{t_{i+1}} \|X'''(t)\|^2 dt \to \min$$

$$w_s, w_{3,i} \in [0, 1]; \quad \sum_{i=1}^{n} w_{3,i} = 1$$

B-Spline Curves

This principle is applied to quintic B-Spline curves:

$$X(t) = \sum_{i=1}^{4n+2} d_i N_i^5(t)$$

Knot Vector:

$$\{\underbrace{t_1, \ldots, t_1}_{6\ times}, \underbrace{t_2, t_2, t_2, t_2}_{4\ times}, \ldots, \underbrace{t_n, t_n, t_n, t_n}_{4\ times}, \underbrace{t_{n+1}, \ldots, t_{n+1}}_{6\ times}\}.$$

The control points d_i are used as input parameter for the variational computation and one obtains a three-step algorithm:

B-Spline Curves

1) **Least Squares Fitting**:

$$SL := \sum_{j=2}^{n_p} w_{pj}(X(t_{pj}) - P_j)^2 \to \min$$

in B-Spline representation:

$$\sum_{j=1}^{n_p} w_{pj} \left(\sum_{i=1}^{4n+2} d_i N_i^5 (t_{pj} - P_j) \right)^2 \to \min$$

$$\sum_{i=1}^{4n+1} d_i \left(2 \sum_{j=1}^{n_p} w_{pj} - N_i^5(t_{pj}) \right) N_r^5(t_{pj}) = 2 \sum_{j=1}^{n_p} w_{pj} P_j N_r^5(t_{pj})$$

B-Spline Curves

2) Automated Smoothing Process:
Smoothing Criterion: $I_3 \rightarrow \min$
The standard variational procedure yields a linear equation system (see [Hagen, Santarelli, 1992]).

B-Spline Curves

3) Merging:

We now combine these two steps to an overall concept:

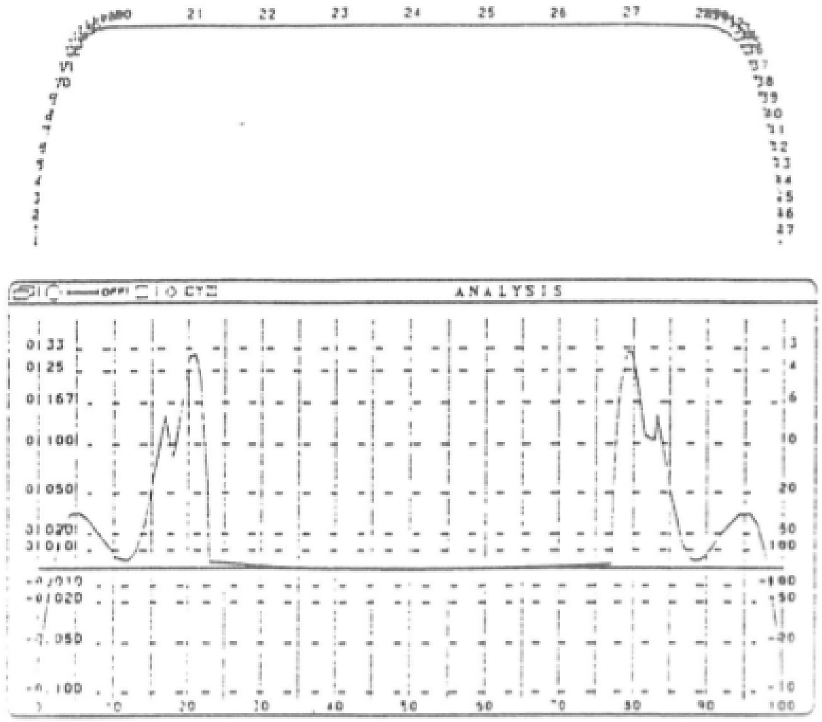
$$(1 - w_s)SL + w_s I_3 \to \min$$

which again yields an overall equation system (see [Hagen, Santarelli, 1992]).

I_3 is defined the same as for the polynomial Bézier curve.

B-Spline Curves

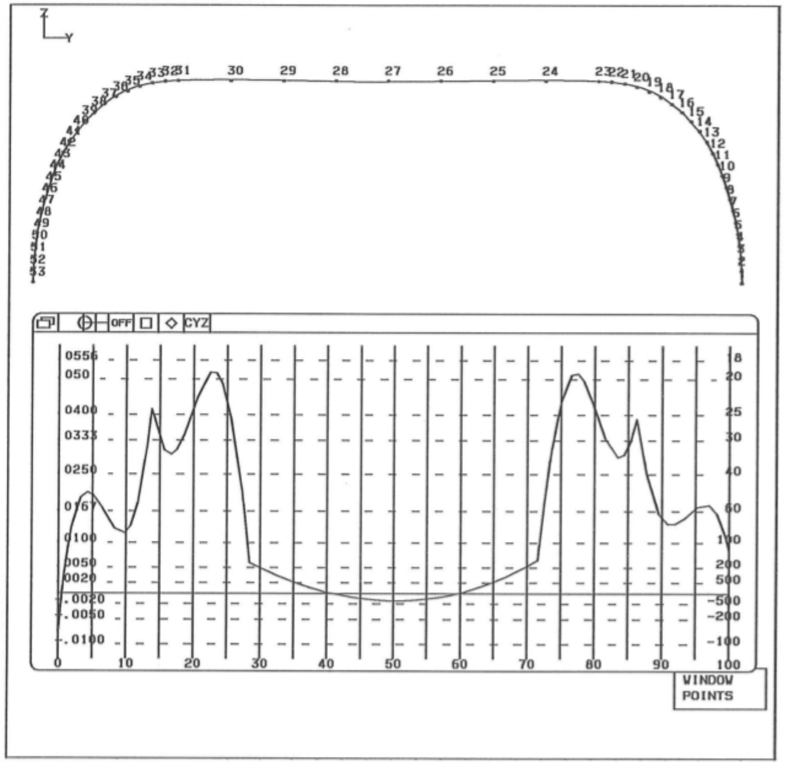
Application:



B-Spline Curves

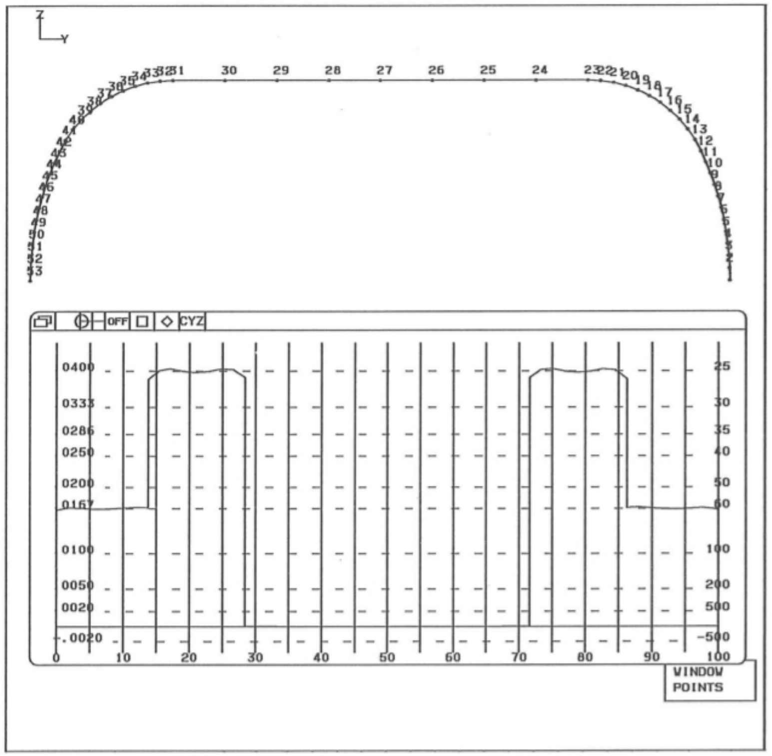
Application:

Using the above algorithm, the curvature behavior can be substantially enhanced:



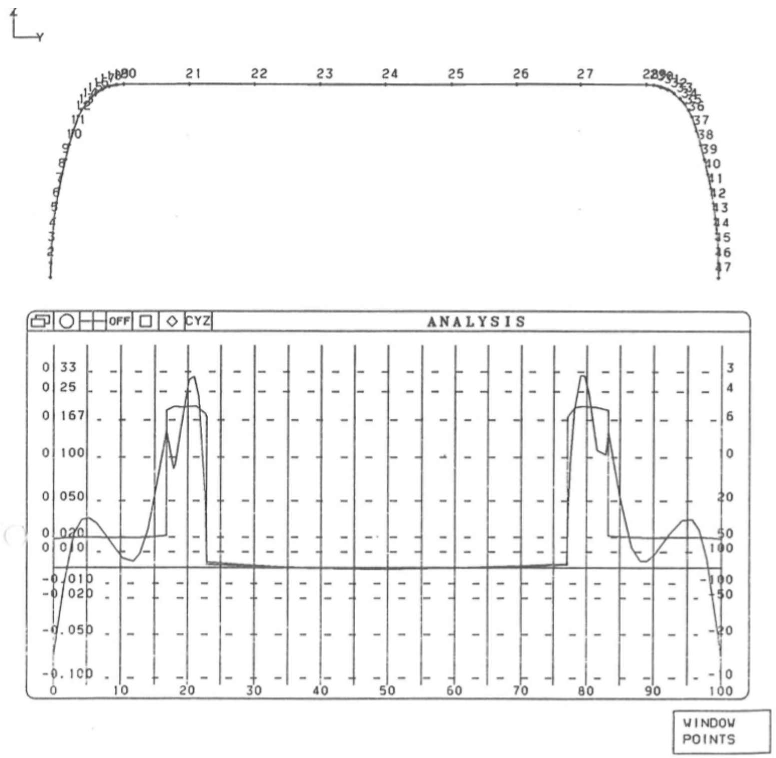
B-Spline Curves

Application:



B-Spline Curves

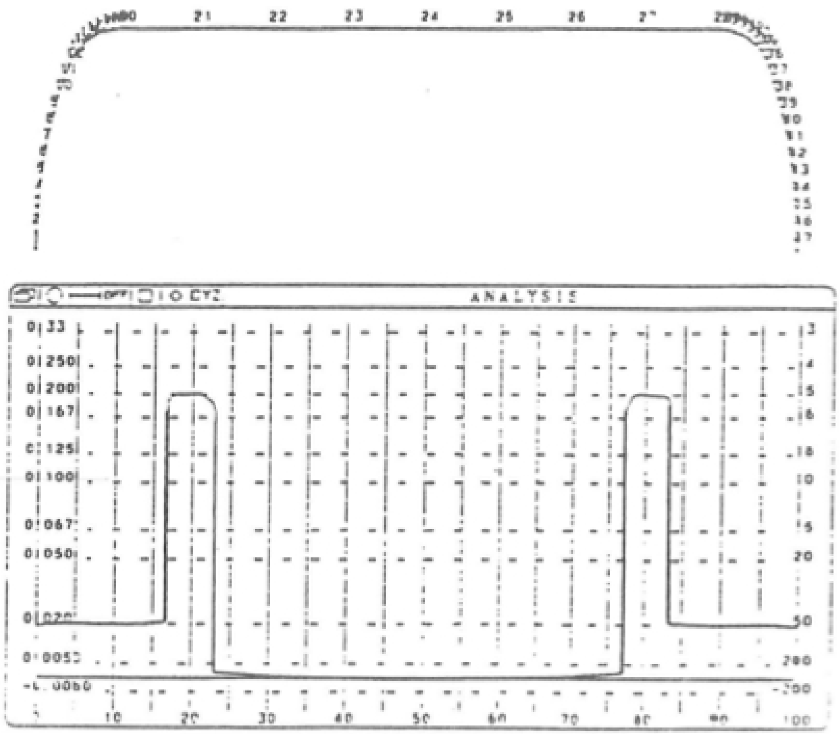
Application:



B-Spline Curves

Application:

Using the above algorithm on the same input data:



B-Spline Surfaces

In this section, we leave the "classical" surface design and:

- construct a smooth curve network
- and include patches into the network

thus proceeding to a more "direct" construction method.

B-Spline Surfaces

In many applications, one has an "old prototype" or a progenitor model from which digitalized point data is scanned. These points do not have to be interpolated, they are merely serve as "points of reference" for the surface design.
A combination of Least Squares fitting and jerk minimization along the parameter lines serves as the mathematical model.

B-Spline Surfaces

$$
\begin{aligned}
&(1 - w_s)\left\{ \sum_{k=1}^{n_p} w_{pk} [X(u_k, w_k) - P_k]^2 \right\} \\
&+ w_s \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} w_{3,u} \int_{v_j}^{v_{j+1}} \int_{u_i}^{u_{i+1}} w_{3,u_{ij}} \left\| \frac{\partial^3 X(u,v)}{\partial u^3} \right\|^2 du\, dv \right. \\
&\left. + w_{3,v} \int_{v_j}^{v_{j+1}} \int_{u_i}^{u_{i+1}} w_{3,v_{ij}} \left\| \frac{\partial^3 X(u,v)}{\partial v^3} \right\|^2 du\, dv \right\} \\
&\longrightarrow \min
\end{aligned}
$$

B-Spline Surfaces

where

$$w_s, w_{3,u}, w_{3,v}, w_{3,u_{ij}}, w_{3,vij} \in [0,1]$$

and

$$\sum_{i=1}^{n}\sum_{j=1}^{n} w_{3,u_{ij}} = 1; \quad \sum_{i=1}^{n}\sum_{j=1}^{n} w_{3,v_{ij}} = 1$$

B-Spline Surfaces

This variational principle is now applied to quintic B-Spline surfaces:

$$X(u,v) = \sum_{i=1}^{4n+2} \sum_{j=1}^{4n+2} d_{ij} N_i^5(u) N_j^5(v)$$

Knot Vector:

$$\{\underbrace{u_1,\ldots,u_1}_{6\ times}, \underbrace{u_2,u_2,u_2,u_2}_{4\ times},\ldots,\underbrace{u_n,u_n,u_n,u_n}_{4\ times},\underbrace{u_{n+1},\ldots,u_{n+1}}_{6\ times}\}$$

$\{v\cdots\}$ analoguously, i.e. the surface is $\mathcal{G}^1$-continuous.

B-Spline Surfaces

As an input for the variational process, we again use the control points $\{d_{ij}\}$. Again, this yields a three-step algorithm:

1) Least Squares Fitting:

$$SL := \sum_{k=1}^{n_p} w_{pk}[X(u_k, v_k) - P_k]^2 \to \min$$

in B-Spline representation:

$$\sum_{k=1}^{n_p} w_{pk} \left(\sum_{l=1}^{4n+2} \sum_{r=1}^{4n+2} d_{lr} N_l^5(u_k) N_r^5(v_k) - P_k \right)^2 \to \min$$

B-Spline Surfaces

The neccessary condition $\frac{\partial SL}{\partial d_{ij}} = 0$ yields the following linear equation system:

$$\sum_{l=1}^{4n+2} \sum_{r=1}^{4n+2} \left\{ \sum_{k=1}^{n_p} w_{pk} N_l^5(u_k) N_r^5(v_k) N_i^5(u_k) N_j^5(v_k) \right) d_{lr}$$

$$= \; 2 \sum_{k=1}^{n_p} w_{pk} P_k N_i^5(u_k) N_j^5(v_k)$$

B-Spline Surfaces

2) Automated Smoothing Process:

Smoothing Criterion:

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\int_{v_j}^{v_{j+1}}\int_{u_i}^{u_{i+1}} w_{3,u} w_{3,u_{ij}} \left\| \frac{\partial^3 X(u,v)}{\partial u^3} \right\|^2 + w_{3,v} w_{3,v_{ij}} \left\| \frac{\partial^3 X(u,v)}{\partial v^3} \right\|^2 du\, dv \longrightarrow \min$$

The standard variation procedure yields a linear equation system (see [Hagen, Santarelli: Variational Design of Smooth B-Spline Surfaces, 1992]).

B-Spline Surfaces

3) Merging:

We now combine the three steps into an overall concept:

$$(1 - w_s)A + w_s B = 0$$

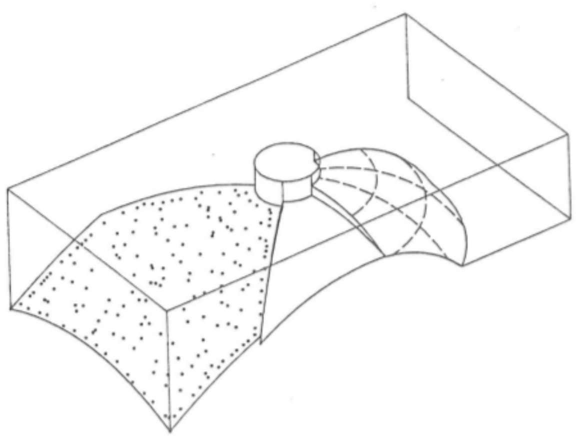
A symbolizes the equations from the linear equation system obtained in the least squares fitting step and b represents the equations emerging from the smoothing criterion.

B-Spline Surfaces

Application:

We now use the procedure to model the reflector surface of a car's headlight:

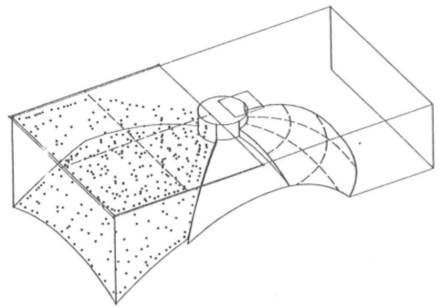
1) Digitalization:



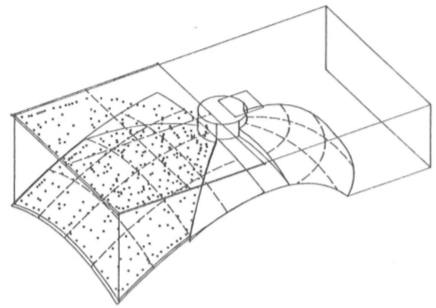
B-Spline Surfaces

Application:

2) Parameterization:



3) Variational Surface Design:



Variational Design of Rational Bézier Surfaces

Rationale Bézier Surface:

$$X(u,v) = \frac{\sum_{i=0}^{n}\sum_{j=0}^{m} w_{ij} B_i^n(u) B_j^m(v) P_{ij}}{\sum_{i=0}^{n}\sum_{j=0}^{m} w_{ij} B_i^n(u) B_j^m(v)}$$

Smoothing Criterion:

$$\int_S (\kappa_1^2 + \kappa_2^2)\, dS \rightarrow \min$$

Variational Design of Rational Bézier Surfaces

$$\int_S (\kappa_1^2 + \kappa_2^2) dS = \int_{u_1}^{u_2} \int_{v_1}^{v_2} \frac{(g_{11} h_{22} - 2 g_{12} h_{12} + g_{22} h_{22}) - 2gh}{g^2} \sqrt{g}\, du\, dv$$

We approximate this integral using the *trapezoidal rule*:

$$\int_{u_1}^{u_2} \int_{v_1}^{v_2} f(u, v) du\, dv \approx \frac{(u_2 - u_1)(v_2 - v_1)}{4}$$
$$\cdot\ (f(u_1, v_1) + f(u_2, v_1) + f(u_1, v_2) + f(u_2, v_2))$$

Variational Design of Rational Bézier Surfaces

In the case of a bicubic rational Bézier surface, the usual variation steps yield a *linear (!)* equation system for the "inner" weights w_{11}, w_{12}, w_{21}, and w_{22}, that has the following unique solution:

$$\omega_{11} = \frac{\omega_{10}^2\omega_{20}\langle b_{10}-b_{00}, b_{10}-b_{00}\rangle\langle b_{20}-b_{00}, N_{00}\rangle + \omega_{01}^2\omega_{02}\langle b_{01}-b_{00}, b_{01}-b_{00}\rangle\langle b_{02}-b_{00}, N_{00}\rangle}{\omega_{10}\omega_{01}\left(\langle b_{10}-b_{00}, b_{10}-b_{00}\rangle\langle b_{01}-b_{00}, b_{01}-b_{00}\rangle + \langle b_{10}-b_{00}, b_{01}-b_{00}\rangle^2\right)} \times \frac{\langle b_{10}-b_{00}, b_{01}-b_{00}\rangle}{\langle b_{11}-b_{00}, N_{00}\rangle}$$

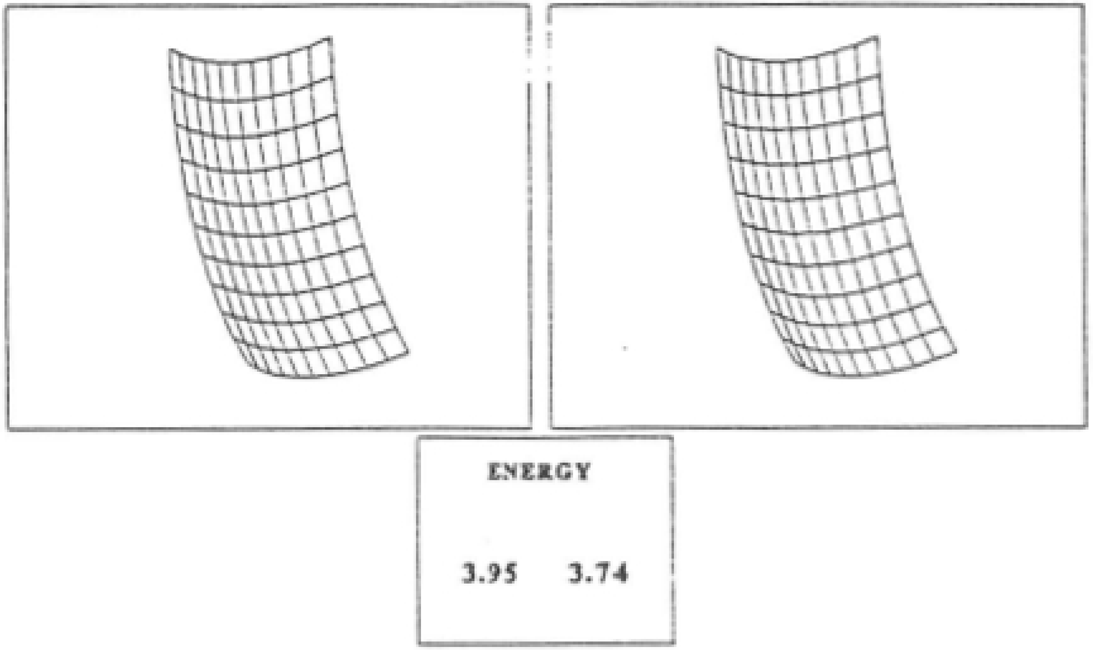
$$\omega_{12} = \frac{\omega_{13}^2\omega_{23}\langle b_{13}-b_{03}, b_{13}-b_{03}\rangle\langle b_{23}-b_{03}, N_{03}\rangle + \omega_{02}^2\omega_{01}\langle b_{02}-b_{03}, b_{02}-b_{03}\rangle\langle b_{01}-b_{03}, N_{03}\rangle}{\omega_{13}\omega_{02}\left(\langle b_{13}-b_{03}, b_{13}-b_{03}\rangle\langle b_{02}-b_{03}, b_{02}-b_{03}\rangle + \langle b_{13}-b_{03}, b_{02}-b_{03}\rangle^2\right)} \times \frac{\langle b_{13}-b_{03}, b_{02}-b_{03}\rangle}{\langle b_{12}-b_{03}, N_{03}\rangle}$$

$$\omega_{21} = \frac{\omega_{20}^2\omega_{10}\langle b_{20}-b_{30}, b_{20}-b_{30}\rangle\langle b_{10}-b_{30}, N_{30}\rangle + \omega_{31}^2\omega_{32}\langle b_{31}-b_{30}, b_{31}-b_{30}\rangle\langle b_{32}-b_{30}, N_{30}\rangle}{\omega_{20}\omega_{31}\left(\langle b_{20}-b_{30}, b_{20}-b_{30}\rangle\langle b_{31}-b_{30}, b_{31}-b_{30}\rangle + \langle b_{20}-b_{30}, b_{31}-b_{30}\rangle^2\right)} \times \frac{\langle b_{20}-b_{30}, b_{31}-b_{30}\rangle}{\langle b_{21}-b_{30}, N_{30}\rangle}$$

$$\omega_{22} = \frac{\omega_{23}^2\omega_{13}\langle b_{23}-b_{33}, b_{23}-b_{33}\rangle\langle b_{13}-b_{33}, N_{33}\rangle + \omega_{32}^2\omega_{31}\langle b_{32}-b_{33}, b_{32}-b_{33}\rangle\langle b_{31}-b_{33}, N_{33}\rangle}{\omega_{23}\omega_{32}\left(\langle b_{23}-b_{33}, b_{23}-b_{33}\rangle\langle b_{32}-b_{33}, b_{32}-b_{33}\rangle + \langle b_{32}-b_{33}, b_{23}-b_{33}\rangle^2\right)} \times \frac{\langle b_{32}-b_{33}, b_{23}-b_{33}\rangle}{\langle b_{22}-b_{33}, N_{33}\rangle}$$

Variational Design of Rational Bézier Surfaces

With this method, we can sometimes enhance the polynomial situation even further:



turbine blade

Variational Design of Rational Bézier Surfaces

Nowacki-Functional for Curve and Network Optimization:

$$\sum_{i=1}^{N-1} \rho_i \int_{t_i}^{t_{i+1}} \left\| \frac{d^m}{dt^m} C(t) \right\|^2$$
$$+ \sum_{i=1}^{N} \sum_{j=1}^{M} \omega_i^j \left\| \frac{d^j}{dt^j} C(t) \Big|_{t=t_i} - P_i^{(j)} \right\|^2 \rightarrow \min$$

where the ρ_i and ω_i are weights as before, m is the order for the least squares fit, n the order for the curve approximation of curve C with parameter t, and the P_i points on the curve.

Variational Design of Rational Bézier Surfaces

This is an approximative energy maximization combined with Least Squares fitting up to the order of m where the t_i separate the segments.

Variational Design of Rational Bézier Surfaces

Generalization to Networks (with weights $\rho_{i,j}$ and $\omega_{i,j}$):

$$J_{NET} = \sum_{i=1}^{M-1}\sum_{j=1}^{N}\rho_{i,j}^{\tilde{u}}\int_{\tilde{u}_{i,j}}^{\tilde{u}_{i+1,j}}\left\|\frac{d^m}{d\tilde{u}^m}\underline{c}_{i,j}(\tilde{u})\right\|^2 d\tilde{u}$$

$$+\sum_{i=1}^{M}\sum_{j=1}^{N-1}\rho_{i,j}^{\tilde{v}}\int_{\tilde{v}_{i,j}}^{\tilde{v}_{i,j+1}}\left\|\frac{d^m}{d\tilde{v}^m}\underline{d}_{i,j}(\tilde{v})\right\|^2 d\tilde{v}$$

$$+\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{k=1}^{m}\omega_{i,j}^{\tilde{u}^k}\left\|\frac{d^k}{d\tilde{u}^k}\underline{c}_{i,j}(\tilde{u})\Big|_{\tilde{u}=\tilde{u}_{i,j}} - \underline{Q}_{\tilde{u}_{i,j}}^{(k)}\right\|^2$$

$$+\sum_{i=1}^{M}\sum_{j=1}^{N}\sum_{k=1}^{m}\omega_{i,j}^{\tilde{v}^k}\left\|\frac{d^k}{d\tilde{v}^k}\underline{d}_{i,j}(\tilde{v})\Big|_{\tilde{v}=\tilde{v}_{i,j}} - \underline{Q}_{\tilde{v}_{i,j}}^{(k)}\right\|^2$$

with the following smoothing criteria:

- the smoothing criterion of order m for curves in $\tilde{u}$-direction:

$$\sum_{i=1}^{M-1} \sum_{j=1}^{N} \rho_{i,j}^{\tilde{u}} \int_{\tilde{u}_{i,j}}^{\tilde{u}_{i+1,j}} \left\| \frac{d^m}{d\tilde{u}^m} \underline{c}_{i,j}(\tilde{u}) \right\|^2 d\tilde{u}$$

- the smoothing criterion of order m for curves in $\tilde{v}$-direction:

$$\sum_{i=1}^{M} \sum_{j=1}^{N-1} \rho_{i,j}^{\tilde{v}} \int_{\tilde{v}_{i,j}}^{\tilde{v}_{i,j+1}} \left\| \frac{d^m}{d\tilde{v}^m} \underline{d}_{i,j}(\tilde{v}) \right\|^2 d\tilde{v}$$

Variational Design of Rational Bézier Surfaces

The latter ones of the introduced methods assume the data to be largely unstructured. Instead, in presence of a certain "basic structure", in general one uses a two-step procedure:

1. construction of a curve network
2. inclusion of the surface elements (patches)

Variational Design of Rational Bézier Surfaces

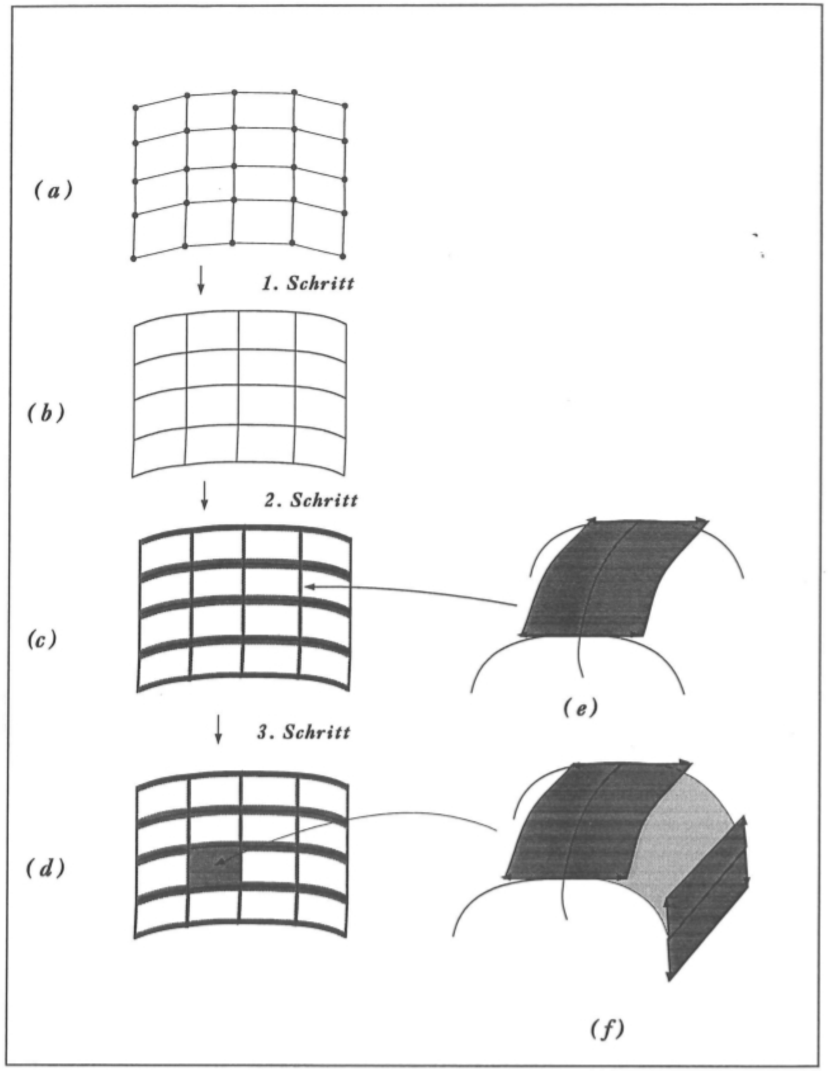
With the same basic idea, higher flexibility can be achieved by a three-step procedure [Nowacki, 1994]:

1. construction of a curve network interpolating the given point set
2. construction of a network of tangent strips that is continuous in the tangential planes and compatible to the surface twist
3. local construction of the surface elements (patches) by interpolation of the boundary curves and tangent strips

Variational Design of Rational Bézier Surfaces

This princlipe is applicable to Bézier- as well as B-Spline surfaces, rational and polynomial.
According to the structure, this seems natural for Bézier surfaces.

Variational Design of Rational Bézier Surfaces



Variational Design of Rational Bézier Surfaces

Another Variational Design Principle:
Veltkamp-Wesselink:

$$\text{Ext} + \int f(t)\|X'(t)\|^2 + g(t)\|X''(t)\|^2 + h(t)\|X'''(t)\|^2 dt \rightarrow \min$$

where f, g, and h are weight functions.

Variational Design of Rational Bézier Surfaces

where Ext =

(plane attractor) $$\int \tilde{f}(t)(\langle h, X(t)\rangle - a)^2 dt$$

or

(curve attractor) $$\int \tilde{f}(t)\|X(t) - f(t)\|^2 dt$$

or

(point attractor) Least Squares Fitting

or

(director) $$\int \tilde{f}(t)\|[X'(r), r]\| dt$$

with weight function $\tilde{f}$.

Variational Design of Rational Bézier Surfaces

Cost Functional in Variational Design:

exact	approximation	geometry	physics
$\int ds$	$\int \|X_u\|^2 + \|X_w\|^2 \, dudw$	area	membrane energy
$\int_s x_1^2 + x_2^2 ds$	$\int \|X_{uu}\|^2 + 2\,\|X_{uw}\|^2 + \|X_{ww}\|^2 \, dudw$	curvature	thin plate energy
$\int \left(\frac{\partial x_1}{\partial e_1}\right)^2 + \left(\frac{\partial x_2}{\partial e_2}\right)^2 ds$	$\int \|X_{uuu}\|^2 + \|X_{www}\|^2 \; dudw$	variation of curvature	jerk