Geometric Modelling Summer 2018

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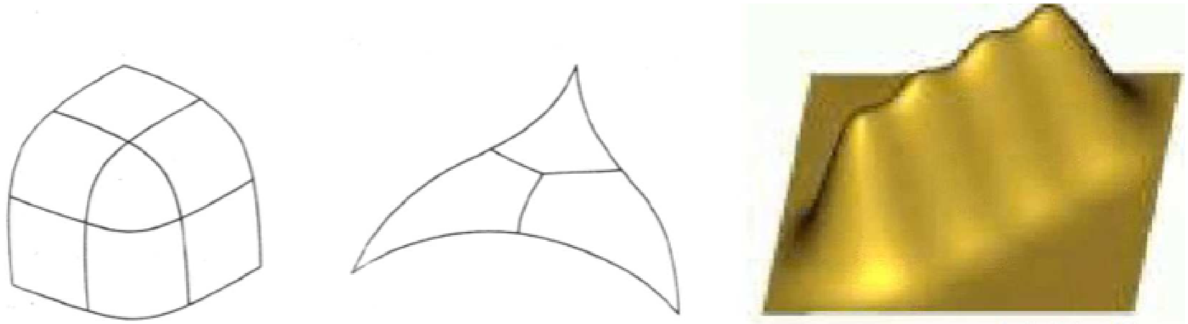
http://hci.uni-kl.de/teaching/geometric-modelling-ss2018

Interpolating Triangle Patches

Motivation

In many applications are situations where it makes sense to work with triangular interpolations.



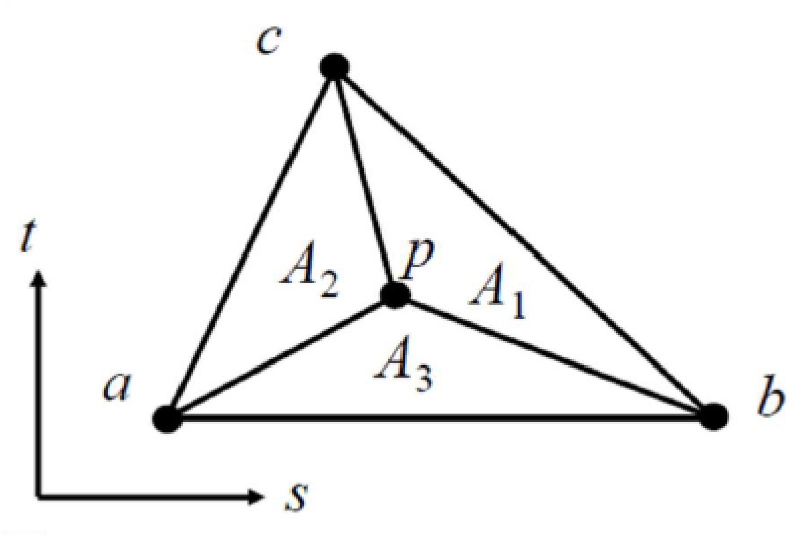
Quelle: Peters, Shiue

The example above is typical in so far that large extends can be rendered using quadriliteral patches but in regions of hich curvature change triangular patches are advantageous. areas of high curvature change but triangular patches are required.

Recapitulation: Bézier Triangle Patches

Barycentric Coordinates:

- domain: any triangle *abc* in the plane
- an arbitrary point *p* in the plane can be defined uniquely in **barycentric coordinates** as follows:



$$p = (u, v, w),$$

$$u = \frac{A_1}{A}, v = \frac{A_2}{A}, w = \frac{A_3}{A}$$

$$A_1 = \|(c-b)\times(p-b)\|/2$$

$$A_2 = \|(b-a)\times(p-a)\|/2$$

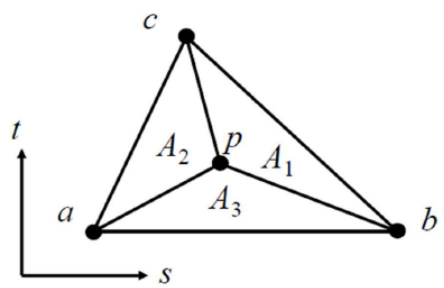
$$A_3 = \|(a-c)\times(p-c)\|/2$$

$$A_4 = \|(c-b)\times(a-b)\|/2$$

Recapitulation: Bézier Triangle Patches

Properties:

- if p is outside the triangle abc, some A_i are negative
- the points a, b, and c have the varicentric coordinates $v_1 = (1, 0, 0)$, $v_2 = (0, 1, 0)$, and $v_3 = (0, 0, 1)$ respectively
- at the edges of the triangle, one of the barycentric coordinates is zero
- inside the triangle, u, v, and w are positive
- barycentric coordinates of a vector d add to zero: $d = (u_1, v_1, w_1) - (u_2, v_2, w_2)$

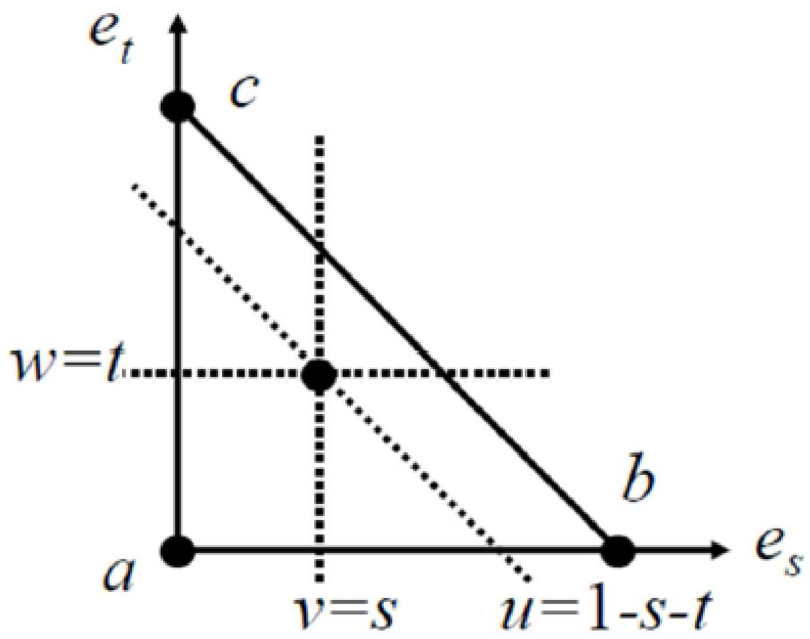


Recapitulation: Bézier Triangle Patches

Properties:

- with the coordinate system $e_s = b - a$ and $e_t = c - a$, one gets a simple conversion to Cartesian coordinates:

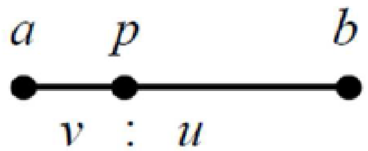
$$u = 1 - s - t; \quad v = s; \quad w = t$$



Recapitulation: Bézier Triangle Patches

General Barycentric Coordinates:

- a **simplex** is defined by $n+1$ (affinely independent) points in $\mathbb{R}^n$, e.g. line segment, triangle, tetrahedron, ...
- barycentric coordinates can be defined for any n-dimensional simplex:



- linear ratios of vectors along lines can also be described using barycentric coordinates:
  instead of t and $(1-t)$ one uses the barycentric coordinates relative to the points a and b:

$$u = (1-t), \quad v = t$$

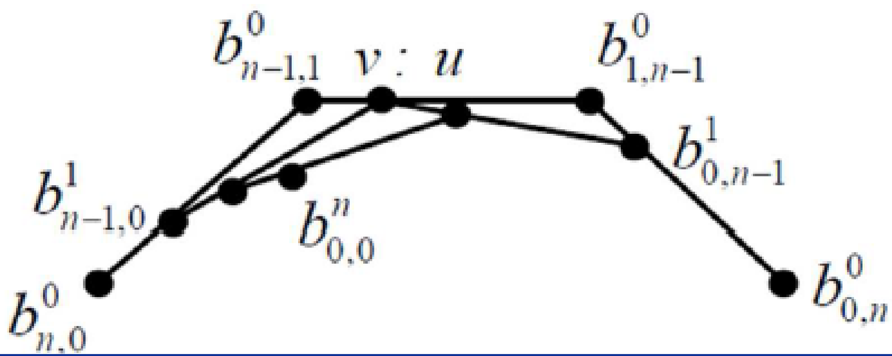
Recapitulation: Bézier Triangle Patches

De Casteljau Algorithm:

- de Casteljau's algorithm for curves can be written in barycentric coordinates as follows:

$$b^0_{i,n-i} := b_i \qquad (i = 0, \ldots, n)$$
$$b^r_{i,j} := u \cdot b^{r-1}_{i+1,j} + v \cdot b^{r-1}_{i,j+1} \quad (r = 1, \ldots, n; \quad i + j = n - r)$$
$$f(u, v) = n^n_{0,0}$$

- the *univariate* Bernstein polynomials then have the form $B^n_{i,j}(u, v) = \frac{n!}{i!j!} u^i v^i$



Recapitulation: Bézier Triangle Patches

Trick:

$$1 = (u + v + w)^n = \sum_{\substack{i,j,k=0,\dots,n \\ i+j+k=n}} \frac{n!}{i!j!k!} u^i v^j w^k$$

Definition: Bivariate Bernstein Polynomials

The polynomials

$$B^n_{i,j,k}(u, v, w) = \frac{n!}{i!j!k!} u^i v^j w^k$$

are called **bivariate Bernstein polynomials** of degree n.

Recapitulation: Bézier Triangle Patches

Properties of the Beinstein Polynomials:

- the Bernstein polynomials are non-negative and sum up to one:

a) $B^n_{i,j,k}(u,v,w) \geq 0$ for $u,v,w \geq 0, \quad (u+v+w=1)$

b) $$\sum_{\substack{i,j,k=0,\ldots,n\\ i+j+k=n}} \frac{n!}{i!j!k!} u^i v^j w^k = (u+v+w)^n = 1$$

- recursion formula:

$$\begin{aligned} B^n_{i,j,k}(u,v,w) = & u \cdot B^{n-1}_{i-1,j,k}(u,v,w) \\ & + v \cdot B^{n-1}_{i,j-1,k}(u,v,w) \\ & + w \cdot B^{n-1}_{i,j,k-1}(u,v,w) \end{aligned}$$

Recapitulation: Bézier Triangle Patches

Definition: Bézier Triangle Surface

A surface of the form

$$f(u,v,w) = \sum_{\substack{i,j,k=0,\dots,n\\ i+j+k=n}} b_{i,j,k} B^n_{i,j,k}(u,v,w)$$

is called a **Bézier triangle surface**.

A surface of polynomial degree n is given by $\frac{(n+1)(n+2)}{2}$ Bézier points $b_{i,j,k} = 0,\dots n,\ i+j+k=n$ on a regular triangle grid.

Recapitulation: Bézier Triangle Patches

Examples:

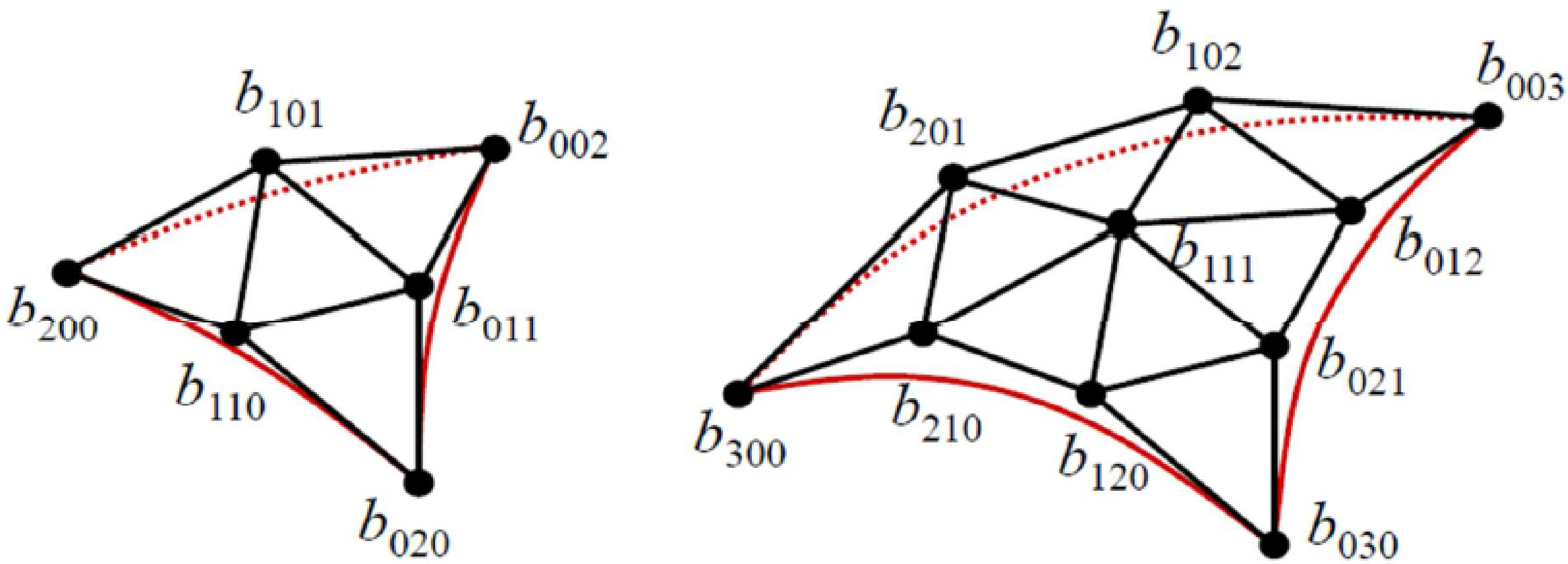


Figure: Examples for Bézier triangle surfaces. left: $n = 2$ (quadratic). right: $n = 3$ (cubic)

Recapitulation: Bézier Triangle Patches

De Casteljau Algorithm:

To evaluate a Bézier triangle surface $f(u, v, w)$ in the barycentric coordinates 8u, v, w), the de Casteljau algorithm is used:

$$b^0_{i,j,k} := b_{i,j,k} \quad (i, j, k \geq 0; \quad i + j + k = n)$$

$$b^r_{i,j,k} := u \cdot b^{r-1}_{i+1,j,k} + v \cdot b^{r-1}_{i,j+1,k} + w \cdot b^{r-1}_{i,j,k+1}$$

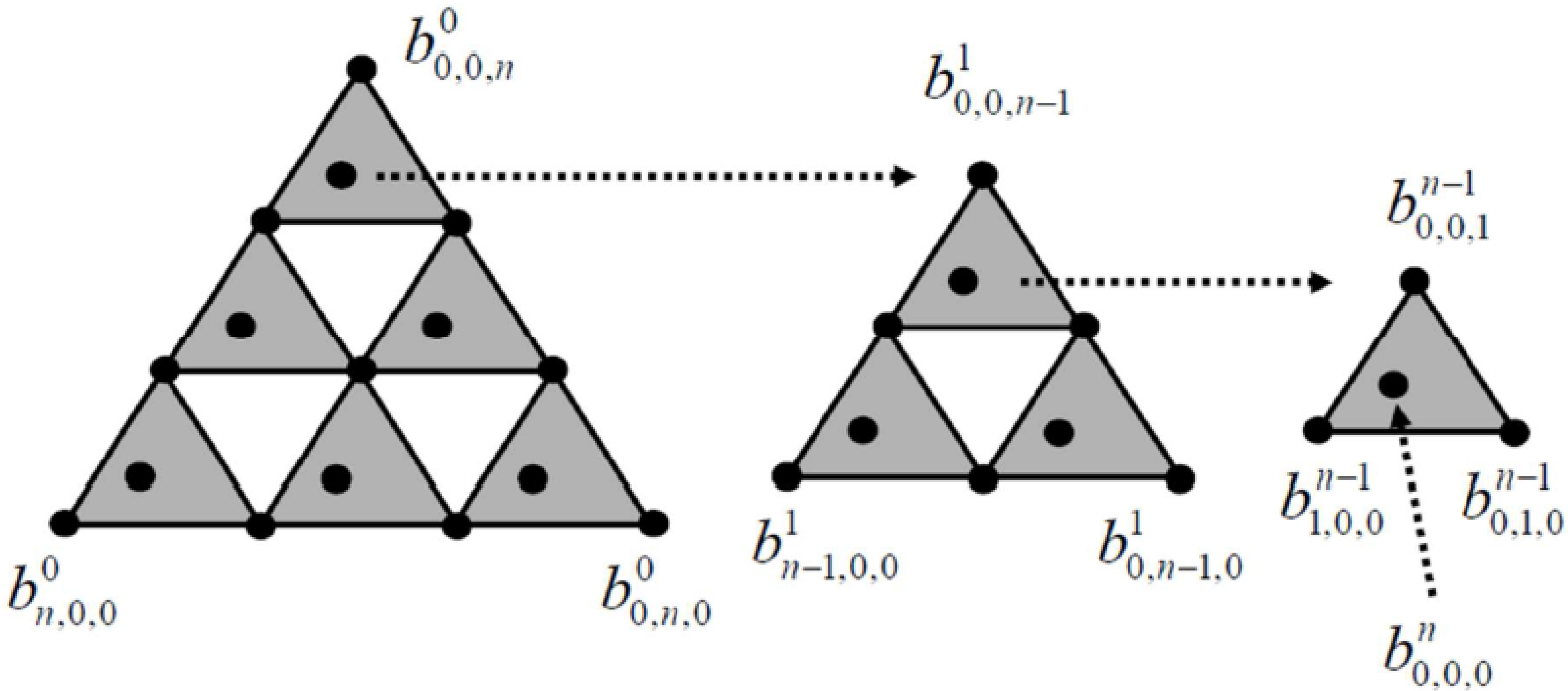
$$(r = 1, \dots, n; \quad i + j + k = n - r)$$

$$f(u, v, w) = b^n_{0,0,0}$$

Recapitulation: Bézier Triangle Patches

De Casteljau Algorithm:

This corresponds to the following pyramid-scheme:



Recapitulation: Bézier Triangle Patches

Properties of Bézier Triangle Surfaces:

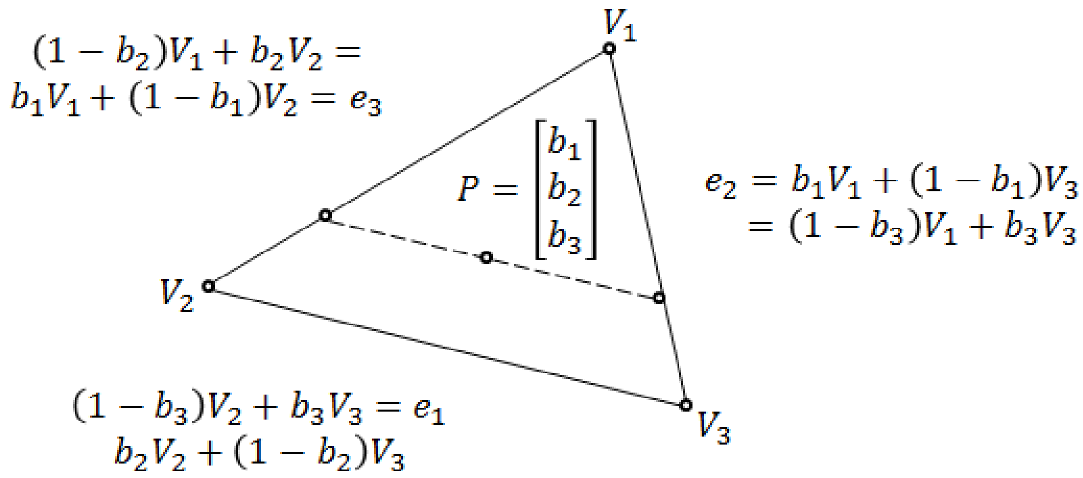
- the surface lies completely *inside the convex hull* of the Bézier net
- the *boundary curves* are Bézier curves
- every surface curve along a line is a polynomial of degree n
  $\rightarrow$ the surface has rotational symmetry
  $\rightarrow$ no two directions are preferred (like for tensor product surfaces)

C^0-Interpolation Scheme

The first method for triangular interpolations was specified by Barnhill, Birkhoff and Gordon. We give here a generalized version for general triangles using barycentric coordinates:

C^0-Interpolation Scheme

First, suitable interpolation operators to C^0-data are formed parallel to the edges.



C^0-Interpolation Scheme

$$\begin{aligned} P_1(F) &:= \tfrac{b_3}{1-b_2} F(e_1) + \tfrac{b_1}{1-b_2} F(e_3) \\ P_2(F) &:= \tfrac{b_3}{1-b_1} F(e_2) + \tfrac{b_2}{1-b_1} F(e_3) \\ P_3(F) &:= \tfrac{b_1}{1-b_3} F(e_2) + \tfrac{b_2}{1-b_3} F(e_1) \end{aligned}$$

Set b_2 in $P_1(F)$, b_1 in $P_2(F)$ and b_3 in $P_3(F)$ equal zero then in each case two straight arise.
These interpolation projectors are not commutative, that is
$P_i \circ P_j \neq P_j \circ P_i; \quad i \neq j; \quad i, j = 1, 2, 3$

C^0-Interpolation Scheme

Since each projector $P_i(F)$ parallel to one side linearly interpolates between the values of the other two sides, this projectors are referred to as **lofting interpolants**.
With this three projectors $P_1(F)$, $P_2(F)$ and $P_3(F)$ a symmetrical, all triangle sides equally treating interpolation scheme can be specified.

C^0-Interpolation Scheme

Theorem: C^0- Interpolation (Barnhill-Birkhoff-Gordon-1973)

(a) The operator $Q(F) := \frac{1}{2}\{P_1(F) + P_2(F) + P_3(F) - P_1 \circ P_2 \circ P_3(F)\}$ is idempotent and linear and thus a projector.

(b) The projector $Q(F)$ interpolates the edge ∂T of the triangle, that is $Q(F)|_{\partial T = F}|_{\partial T}$

(c) $L := P_1 \circ P_2 \circ P_3 = P_i \circ P_j \circ P_k; \quad i \neq j \neq k \neq i; \quad i, j, k = 1, 2, 3.$ The order of the projectors is therefore irrelevant.

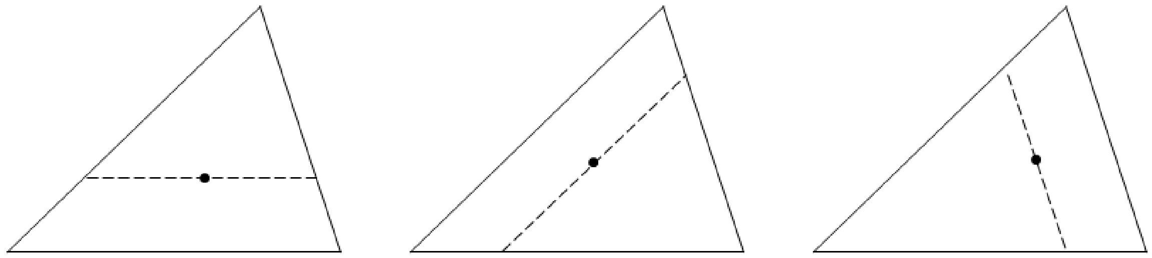
C^0-Interpolation Scheme

In detail is

$$
\begin{aligned}
Q(F) \quad = \quad & \tfrac{1}{2}\{\tfrac{b_3}{1-b_2}F(e_1) + \tfrac{b_1}{1-b_2}F(e_3) \\
& + \tfrac{b_3}{1-b_1}F(e_2) + \tfrac{b_2}{1-b_1}F(e_3) \\
& + \tfrac{b_1}{1-b_3}F(e_2) + \tfrac{b_2}{1-b_3}F(e_1)
\end{aligned}
$$

C^0-Interpolation Scheme

Each projector is a linear interpolant to F on each two sides of the triangle T along parallels to the third side.

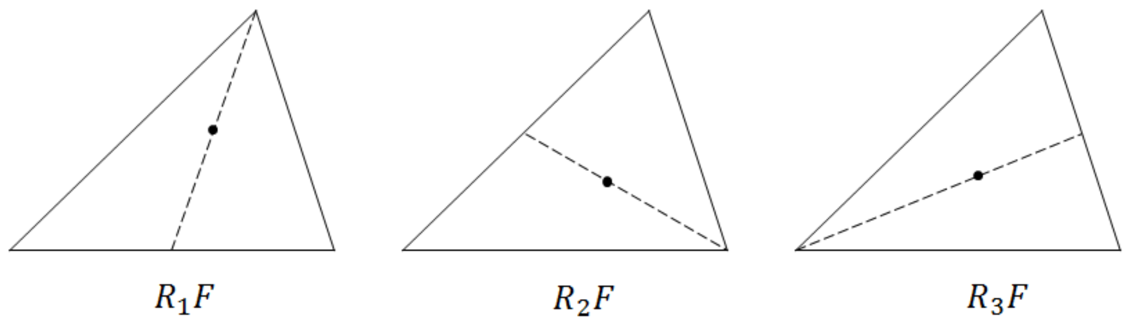


P_1F P_2F P_3F

This scheme is affin invariant, since parallels pass over to parallels in affine mappings.

C^0-Interpolation Scheme

Instead of "parallel" projectors one can use so called "radial" projectors.



The basic idea of this so called **side-vertex-method** is to put together informations to a scheme from one corner and the opposite side of the triangle.
Each point of the side e_i, which is on the opposite of the corner V_i, can be described by:

$$S_i(x,y) = (\frac{x-x_i}{1-b_i}, \frac{y-y_i}{1-b_i}); \quad i = 1,2,3$$

C^0-Interpolation Scheme

The radial projectors can be formulated as:

$$
\begin{aligned}
R_1, F &:= (1-b_1)F\left(\frac{x-x_1 b_1}{1-b_1}, \frac{y-y_1 b_1}{1-b_1}\right) + b_1 F(v_1) \\
R_2, F &:= (1-b_2)F\left(\frac{x-x_2 b_2}{1-b_2}, \frac{y-y_2 b_2}{1-b_2}\right) + b_2 F(v_2) \\
R_3, F &:= (1-b_3)F\left(\frac{x-x_3 b_3}{1-b_3}, \frac{y-y_3 b_3}{1-b_3}\right) + b_3 F(v_3)
\end{aligned}
$$

These radial operators are commutative, means
$R_i \circ R_j = R_j \circ R_i; \quad i = 1, 2, 3$

C^0-Interpolation Scheme

Theorem: C^0-Side-Vertex -Method

(a) The operator $R(F) := R_1 \otimes R_2 \otimes R_3(F)$ is idempotent and linear and thus a projector.

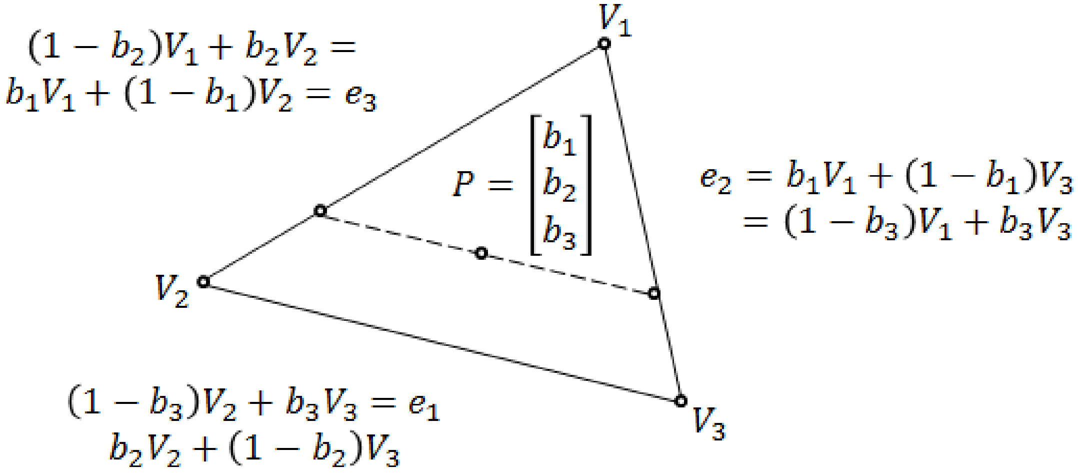
(b) $R(F) = \sum_{i=1}^{3}(1 - b_i)F(S_i) - b_i F(V_i)$

(c) The projector $R(F)$ interpolates the edge ∂T of the triangle, means $R(F)|_{\partial T} = F|_{\partial T}$

(d) The Side-Vertex-Method is affine invariant.

C^1-Boolean-Sum-Interpolation

The C^0-Schemes can be extended to C^1-schemes. We first consider C^1-BBG-Interpolations.



C^1-Boolean-Sum-Interpolation

$$
\begin{aligned}
P_1F \quad &:= \quad H_0\left(\tfrac{b_3}{1-b_1}\right)F(e_2) + H_1\left(\tfrac{b_2}{1-b_1}\right)F(e_3) \\
&\quad + \overline{H}_0\left(\tfrac{b_2}{1-b_1}\right)(1-b_1)\tfrac{\partial F}{\partial e_1}(e_2) + \overline{H}_1\left(\tfrac{b_2}{1-b_1}\right)(1-b_1)\tfrac{\partial F}{\partial e_1}(e_3) \\
P_2F \quad &:= \quad H_0\left(\tfrac{b_3}{1-b_2}\right)F(e_1) + H_1\left(\tfrac{b_1}{1-b_2}\right)F(e_3) \\
&\quad + \overline{H}_0\left(\tfrac{b_3}{1-b_2}\right)(1-b_2)\tfrac{\partial F}{\partial e_2}(e_1) + \overline{H}_1\left(\tfrac{b_1}{1-b_2}\right)(1-b_2)\tfrac{\partial F}{\partial e_2}(e_3) \\
P_3F \quad &:= \quad H_0\left(\tfrac{b_1}{1-b_3}\right)F(e_2) + H_1\left(\tfrac{b_2}{1-b_3}\right)F(e_1) \\
&\quad + \overline{H}_0\left(\tfrac{b_1}{1-b_3}\right)(1-b_3)\tfrac{\partial F}{\partial e_3}(e_2) + \overline{H}_1\left(\tfrac{b_1}{1-b_3}\right)(1-b_3)\tfrac{\partial F}{\partial e_3}(e_1)
\end{aligned}
$$

$H_0, H_1, \overline{H}_0, \overline{H}_1$ are the cubic hermite basefunctions. The formation of boolean sum schemes based on these projectors leads to compatibility conditions.

C^1-Boolean-Sum-Interpolation

Theorem: compatibiliy conditions

$(P_i \otimes P_j)(F); \quad i \neq j; \quad i, j \in 1, 2, 3$ and
$Q(F) := (P_1 + P_2 + P_3 - P_1 \circ P_2 \circ P_3)(F)$ then and only then the boundary curve and the tangent transversely interpolate to the edges, if and only if

$$\frac{\partial^2 F}{\partial e_j \partial e_i}(V_k) = \frac{\partial^2 F}{\partial e_i \partial e_j}(V_k)$$

in which $V_k, k = 1, 2, 3$ are the corners of the triangle and the $\frac{\partial}{\partial e_i}$ represent derivatives in the direction of the triangle sides.

C^1-Boolean-Sum-Interpolation

These compatibiliy conditions aren't always met. Barnhill and Gregory, therefore, developed a theory of how to eliminate commutativity conditions (Vertauschbarkeitsbedingungen).

Theorem

The function $(P_i \otimes P_j)(F) + (P_j \otimes P_k)(R)$ and $(P_i \otimes P_j)(F) + (P_k \otimes P_j)(R)$ in which
$R := F - (P_i \otimes P_j)(F); \quad i \neq j \neq k \neq i; \quad i, j, k = 1, 2, 3$
interpolate $F \in C'(\partial T)$ and the first derivatives on ∂T.

C^1-Boolean-Sum-Interpolation

This result is extendable to the functional class $F \in C^n(\partial T)$ and contains as a special case:

Conclusion

For $F \in C'(\partial T)$ interpolates

$$(P_i \otimes P_j)(F) - \frac{b_i b_j^2 b_k^2}{1 - b_k}\left(\frac{\partial^2}{\partial s_j \partial s_i}(V_k) - \frac{\partial^2 F}{\partial s_i \partial s_j}(V_k)\right);$$

$$i \neq j \neq k \neq i; \quad 1 \leq i, j, k \leq 3$$

F and the first derivatives of F on ∂T.

C^1-Boolean-Sum-Interpolation

A very practicable method is from Gregory

Theorem: compatibiliy corrected BBG-scheme

The boolean sum $(P_3 \otimes (P_1 \otimes P_2))(F)$ interpolates positions and transverse tangents on the edge of any triangle without compatibility conditions.

C^1-Boolean-Sum-Interpolation

The considered C^1-Interpolations use all rational basis functions. One can replace rational basis functions by polynomial functions, such as Barnhill and Gregory have shown. They develop the following design principle for interpolation schemes.

Theorem

$F \in C^1(\partial T)$ and $\partial T = \Gamma_1 \cup \Gamma_2$. P_1 and P_2 are interpolation projectors with $D^\alpha P_i F = D^\alpha F$ on Γ_1 und $D^\alpha P_i F$ is defined on $\Gamma_1 \cup \Gamma_2$ for all $|\alpha| := \alpha_1 + \cdots + \alpha_n \leq N$ with $D^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$

C^1-Boolean-Sum-Interpolation

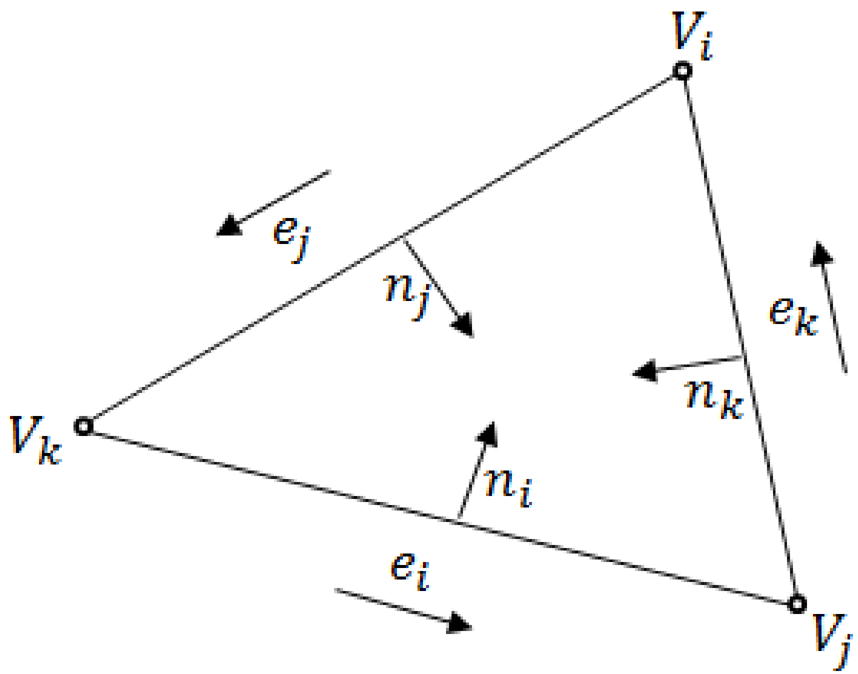
Then it is:

(i) $D^\alpha(P_1 \otimes P_2)F = D^\alpha F$ on Γ_1 for all $|\alpha| \leq N$

(ii) $D^\alpha(P_1 \otimes P_2)F = D^\alpha F$ on $\Gamma_2 \backslash \Gamma_1$ if for all $|\alpha| \leq N$ $D^\alpha P_1 F$ on $\Gamma_2 \backslash \Gamma_1$ is a linearcombination of function values and values of derivatives on Γ_2, which are interpolated by $P_2 F$.

Remark:
The C^0- and C^1-BBG-Interpolations subordinate themselves to this principle. We use this principle now to construate interpolation schemes, which contain polynomial basis functions.

C^1-Boolean-Sum-Interpolation



$$S_i^j := b_j V_j + (1 - b_j) V_k, \quad S_i^k := (1 - b_k) V_j + b_k V_k$$

C^1-Boolean-Sum-Interpolation

We start with

$$\begin{aligned} P_1(F) \quad &:= \quad F(S_2^3) + b_2 \frac{\partial F}{\partial n_2}(S_2^3) + F(S_3^2) + b_3 \frac{\partial F}{\partial n_3}(S_3^2) - F(V_1) \\ &\quad - b_2 \frac{\partial F}{\partial n_2}(V_1) - b_3 \frac{\partial F}{\partial n_3}(V_1) - b_2 b_3 \frac{\partial^2 F}{\partial n_2 \partial n_3}(V_1) \end{aligned}$$

This is the boolean sum of the Taylor projectors:

$$\begin{aligned} T'(F) \quad &:= \quad F(S_2^3) + b_2 \frac{\partial F}{\partial n_2}(S_2^3) \\ T^2(F) \quad &:= \quad F(S_3^2) + b_3 \frac{\partial F}{\partial n_3}(S_3^2) \end{aligned}$$

Here it is: $\frac{\partial b_i}{\partial N_i} = 1$

C^1-Boolean-Sum-Interpolation

That $P_1(F)$ interpolates the function values and the orthogonal transverse tangents on e_2 and e_3 it has to be:

$$\frac{\partial^2 F}{\partial n_2 \partial n_3}(V_1) = \frac{\partial^2 F}{\partial n_3 \partial n_2}(V_1)$$

With the upper Theorem it is now:

$$P_2(F) := \alpha \cdot \left(F(S_2') + b_1 \frac{\partial F}{\partial n_1}(S_2')\right) + \beta \cdot \left(F(S_3') + b_1 \frac{\partial F}{\partial n_1}(S_3')\right)$$

with $\alpha := b_3^2(3 - 2b_3 + 6b_2b_1); \quad \beta := b_2^3(3 - 2b_2 + 6b_3b_1)$

C^1-Boolean-Sum-Interpolation

The blending functions $\alpha(b_1, b_2, b_3)$ and $\beta(b_1, b_2, b_3)$ are a special case ($N = 1$) of:

$$\alpha(b_1, b_2, b_3) := b_1^{N+1} \cdot \sum_{i=0}^{N} \sum_{j=0}^{N} \left(b_2^i b_3^j \frac{(N+i+j)!}{N!i!j!}\right)$$

$$\beta(b_1, b_2, b_3) := b_2^{N+1} \cdot \sum_{i=0}^{N} \sum_{j=0}^{N} \left(b_3^i b_1^j \frac{(N+i+j)!}{N!i!j!}\right)$$

$$\gamma(b_1, b_2, b_3) := b_3^{N+1} \cdot \sum_{i=0}^{N} \sum_{j=0}^{N} \left(b_1^i b_2^j \frac{(N+i+j)!}{N!i!j!}\right)$$

C^1-Boolean-Sum-Interpolation

With a appropriate boolean sum

$$(P_2 \otimes P_1)(F)$$

with $P_2(F)$ and $P_1(F)$ defined above, can a C^1-Interpolation scheme be specified. This scheme can be extended to a C^n-Interpolation scheme, with the necessary commutativity conditions

$$\frac{\partial^{m+n} F}{\partial n_i^m \partial n_j^m} = \frac{\partial^{n+m} F}{\partial n_j^n \partial n_i^m}$$

eliminated by the correcture terms.

Convex Combinations

The essential idea of all schemes of convex combinations is to avoid commutativity conditions from the beginning. The first method of this type was developed by Gregory. We give here a generalized version for general triangles using barycentric coordinates.

Convex Combinations

Theorem: Symmetric Gregory scheme

$$\begin{aligned} P_i(F) \quad := \quad & H_0\left(\frac{b_j}{1-b_i} F(e_j)\right) + H_1\left(\frac{b_j}{1-b_i} F(e_k)\right) \\ & \overline{H}_0\left(\frac{b_j}{1-b_i}(1-b_i)\frac{\partial F}{\partial e_i} F(e_j)\right) + \\ & \overline{H}_1\left(\frac{b_j}{1-b_i}(1-b_i)\frac{\partial F}{\partial e_i} F(e_k)\right) \end{aligned}$$

with $(i, j, k) = (1, 2, 3); (2, 1, 3); (3, 1, 2)$ and $H_0, H_1, \overline{H}_0, \overline{H}_1$ are the cubic hermite basis functions.

Convex Combinations

$$G(F) := \alpha_1 P_1(F) + \alpha_2 P_2(F) + \alpha_3 P_3(F)$$

with $\alpha_1 := b_i^2(3 - 2b_i + 6b_j b_k)$. Then it is:
$G(F)$ interpolates $F \in C'(\partial T)$ and its first derivatives on ∂T.

Convex Combinations

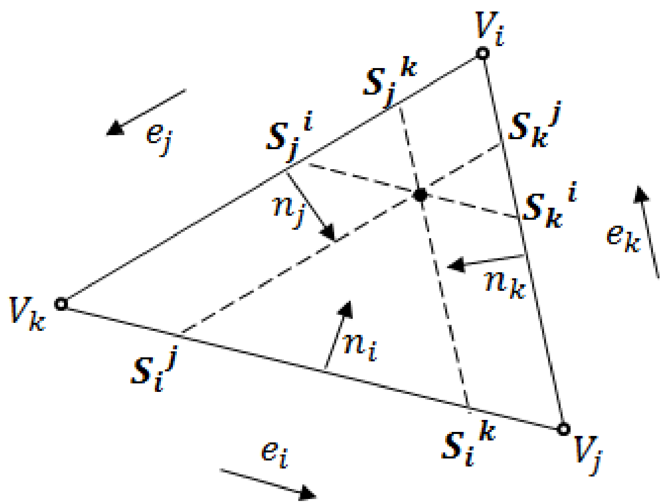
Remarks:

(a) $\sum_{i=1}^{3} \alpha_i = 1; \alpha_i \geq 0; i = 1, 2, 3;$ thats why we say convex combination scheme.

(b) The symmetric gregory scheme doesn't contain mixed derivatives thus its called a "twist-free" interpolation.

Convex Combinations

The symmetric Gregory scheme can be expand into a interpolation scheme that interpolates function values and derivatives up to any order N. This technique is not limited to triangles but may be extended at any dimensional simplices.
Instead of the rational hermite projectors one can also use the projectors builded from the following polynomial taylor projectors.



Convex Combinations

$$
\begin{aligned}
P_i(F) \quad := \quad & F(S_j^k) + b_j \frac{\partial F}{\partial n_j}(S_j^k) + F(S_k^j) + b_k \frac{\partial F}{\partial n_k}(S_k^j) \\
& - F(V_i) - b_j \frac{\partial F}{\partial n_j}(V_i) - b_k \frac{\partial F}{\partial n_k}(V_i) - b_j b_k \frac{\partial^2 F}{\partial n_j \partial n_k}
\end{aligned}
$$

$(i, j, k) = (1, 2, 3); (2, 3, 1); (3, 1, 2)$ and "P_i is not defined along e_i".
It is:

$$P_i(F) = (T_j^k \otimes T_k^j)(F)$$

with the taylor projectors als follows:

$$T_j^k(F) := F(S_j^k) + b_j \frac{\partial F}{\partial n_j}(S_j^k)$$

One can now form a suitable convex combination.

Convex Combinations

Theorem: polynomial symmetric Gregory scheme

$F \in C'(\partial T)$ satisfies the commutative conditions

$$\frac{\partial^2 F}{\partial n_i \partial n_j}(V_k) = \frac{\partial^2 F}{\partial n_j \partial n_i}(V_k)$$

at the corners V_k, where $\frac{\partial}{\partial n_l}$ is the direction derivative perpendicular to the triangular page.

$$PG(F) := \alpha_1 P_1(F) + \alpha_2 P_2(F) + \alpha_3 P_3(F)$$

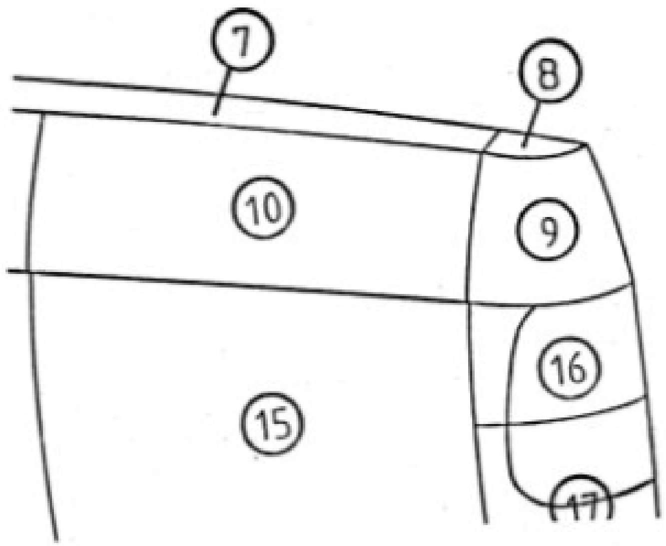
in which $P_i(F)$ is defined above and the blending functions $\alpha_i := b_i^2(3 - 2b_i + 6b_j b_k)$ interpolates $F \in C'(\partial T)$ and its first derivatives on the boundary ∂T.

Convex Combinations

Remark:

- The C^0-variant of the Theorem is: $\alpha_i = b_i$ and $P_i(F) := F(S_j^k) + F(S_k^j) - F(V_i)$.

This triangle patch can easily be fitted in a square pachwork static to the tangential plane.



Convex Combinations

The commutative conditions can be avoided when the following projectors are used:

$$
\begin{aligned}
P_k(F) \quad := \quad & \tfrac{1}{2}(T_i^j \otimes T_j^i)(F) + \tfrac{1}{2}(T_j^i \otimes T_i^j)(F) + \\
& T_k(F - \tfrac{1}{2}(T_i^j \otimes T_j^i)(F) - \tfrac{1}{2}(T_j^i \otimes T_i^j)(F))
\end{aligned}
$$

Convex Combinations

Theorem: compatibility corrected Gregory scheme

The polynomial Gregory scheme $\sum_{k=1}^{3} d_k P_k(F)$, with $P_k(F)$ given from above, interpolates $F \in C'(\partial T)$ and its first derivatives on the boundary ∂T.

Convex Combinations

A certain "expansion" of this triangle interpolation on regular pentagons can be obtained by following considerations:

(1) For any point in the inner of a regular pentagon build the distance λ_i to the 5 pages.

(2) Replace b_i threw λ_i,

(3) and build the convex combinations $P(F) := \sum_{i=1}^{5} \alpha_i P_i(F)$, with $\alpha_i := \frac{\lambda_{i+1}^2 \lambda_i^2 \lambda_{i-1}^2}{\sum_{j=1}^{5} \lambda_{j+1}^2 \lambda_j^2 \lambda_{j-1}^2}$

Convex Combinations

Remark:
This method can be expanded directly to any polygons. However, it is to note that in this method, only in the case of an equilateral triangle, affine invariance is given.
In this special case, the distances, from a point in the interior of the triangle to the triangle sides with the barycentric coordinates of the point, agree.
So far only "two-sided projectors" are used. But you can also with "unilateral projectors", ie operators that interpolate only one side of the triangle, build convex combination schemes.

Convex Combinations

Theorem: Brown-Litte-scheme

$P_iF := F(p_i) + ((x,y)^T - p_i) \cdot \frac{\partial F}{\partial n_i}(p_i); \quad i = 1,2,3$

with $p_i := \frac{<s_i,(x,y)^T - V_j>}{||s_i||^2} \cdot V_k - \frac{<s_i,(x,y)^T - V_k>}{||s_i||^2} \cdot V_j$

$(BL)(F) := \alpha_1 P_1 F + \alpha_2 P_2 F + \alpha_3 P_3 F$

with $\alpha_i := \frac{b_j b_k}{b_1^2 b_2^2 + b_1^2 b_3^2 + b_2^2 b_3^2}$

Convex Combinations

Since now C^1-schemes have been investigated based on parallel projectors. The C^0-Nielson interpolation, which is based on radial projectors, can also be extended to C^1-schemes and beyond to G^1- and G^2-methods. These studies are discussed in the next paragraphs.

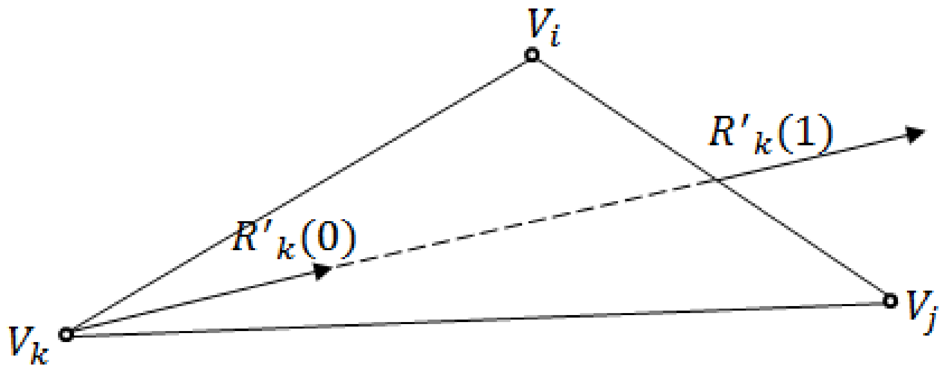
Side-Vertex-Method

To expand the C^0-Nielson interpolation we associate the hermite operator

$$H(g)(t) := H_0(t)g(0) + H_1(t)g(1) + \overline{H}_0(t)g'(0) + \overline{H}_1(t)g'(1)$$

($\{H_0, H_1, \overline{H}_0, \overline{H}_1\}$ is the cubic hermite basis) with the radial operator:

$$R_i(t) := F(ts_i + (1-t)V_i); \quad i = 1, 2, 3$$



Side-Vertex-Method

So we get the following projector:

$$
\begin{aligned}
D_i(F) \quad &:= \quad H_1(b_i)F(V_i) + H_1(1-b_i)F(s_i) \\
&\quad\;\; -\overline{H}_1(b_i)R_i'(0) + \overline{H}_1(1-b_i)R_i'(1) \\
\text{with} & \\
R_i'(0) \quad &= \quad \frac{(x-x_i)F_x(V_i)+(y-y_i)F_y(V_i)}{1-b_i} \\
R_i'(1) \quad &= \quad \frac{(x-x_i)F_x(S_i)+(y-y_i)F_y(S_i)}{1-b_i}
\end{aligned}
$$

Side-Vertex-Method

Theorem: C^1-side-vertex-scheme

Is $F \in C^2_T :=$
$\{F/F \in C'(T); \frac{\partial^{n+m}F}{\partial x^n \partial x^m}|_{S_i} \in C(S_i); n+m=2; i=1,2,3\}$ where
$S_i(x,y) := (\frac{x-x_i b_i}{1-b_i}, \frac{y-y_i b_i}{1-b_i})$ then it holds:

$$D(F) := \frac{b_2^2 b_3^2 D_1(F) + b_1^2 b_3^2 D_2(F) + b_1^2 b_2^2 D_3(F)}{b_2^2 b_3^2 + b_1^2 b_3^2 + b_1^2 b_2^2}$$

is in $C'(T)$ includes and interpolates F and the first derivatives of F on the boundary ∂T.

Side-Vertex-Method

Of course one can build boolean sums. $D_i \circ D_j(F)$ then leads again on commutativity requirements for the mixed derivatives, which can be avoided by the correction terms or using generalized boolean sums.

Side-Vertex-Method

Fundamental for the extension of the side-vertex method in respect to the curvature input possibilities is the following definition of a geometric hermite operator:

$$GH(y) := \sum_{1=0}^{1} H_i(t)y(i) + \overline{H}_i(t)y'(i) + G_i(t)[[y'(i), y''(i)], y'(i)]$$

$y := [0,1] \rightarrow \mathbb{E}^3$ is a parametric space curve.

Side-Vertex-Method

$$
\begin{aligned}
H_0(t) &:= -6t^5 + 15t^4 - 10t^3 + 1 \\
H_1(t) &:= 6t^5 - 15t^4 - 10t^3 \\
G_0(t) &:= \frac{-t^5+3t^4-3t^3+t^2}{2||y'(0)||^4} \\
G_1(t) &:= \frac{t^5-2t^4+t^3}{2||y'(0)||^4} \\
\overline{H}_0(t) &:= -3t^5 + 8t^4 - 6t^3 + t \\
\overline{H}_1(t) &:= -3t^5 + 7t^4 - 4t^3
\end{aligned}
$$

Side-Vertex-Method

The following correlations allow in this context the curvature input:

$$[[y', y''], y'] = ||y'||^4 \cdot k \cdot e_2$$

k is the curve curvature and e_2 the main normal vector of the curve.
The curvature vector $k \cdot e_2$ satisfies the correlation:

$$k \cdot e_2 = k_n \cdot N + k_g \frac{[N, y']}{||y'||}$$

where k is the geodetic curvature, N the surface normal and k_n the normal cutting curvature.
This results in:

$$[[y', y''], y'] = ||y''||^4 (k_n \cdot N + k_g \frac{[N, y']}{||y'||})$$

Side-Vertex-Method

We link now the geometric hermite operator with the radial operator $R_i(t) = F(tS_i + (1-t)V_i)$; $i = 1, 2, 3$ with $S_i := (1 - b_j)(V_k) + b_j V_j$ ad get the following projector:

$$
\begin{aligned}
P_i[F] \quad := \quad & H_0(1-b_i)F(V_i) + H_1(1-b_i)F(S_i) \\
& + \overline{H}_0(1-b_i)R_i'(0) + \overline{H}_1(1-b_i)R_i'(1) \\
& + G_0(1-b_i)[[R_i'(0), R_i''(0)], R_i'(0)] \\
& + G_1(1-b_i)[[R_i'(1), R_i''(1)], R_i'(1)]
\end{aligned}
$$

Side-Vertex-Method

with

$$R_i'(0) = \frac{(x-x_i)F_x(V_i)+(y-y_i)F_y(V_i)}{1-b_i} = R_i'(V_i)$$
$$R_i'(1) = \frac{(x-x_i)F_x(S_i)+(y-y_i)F_y(S_i)}{1-b_i} = R_i'(S_i)$$

and

$$[[R_i'(0), R_i''(0)], R_i'(0)] = ||R_i'(0)||^4(k_N(V_i)N(V_i) + k_g(V_i)\frac{[N(V_i),R_i'(0)]}{||R_i'(0)||})$$
$$[[R_i'(1), R_i''(1)], R_i'(1)] = ||R_i'(1)||^4(k_N(S_i)N(S_i) + k_g(S_i)\frac{[N(S_i),R_i'(1)]}{||R_i'(1)||})$$

Side-Vertex-Method

These projectors can be put together using a convex combination to make a interpolation scheme:

$$P[F] := \frac{b_2^3 b_3^3 P_1[F] + b_1^3 b_3^3 P_2[F] + b_1^3 b_2^3 P_3[F]}{b_2^3 b_3^3 + b_1^3 b_3^3 + b_1^3 b_2^3}$$

Because of the weighting functions $w_i = \frac{b_j^3 b_k^3}{b_2^3 b_3^3 + b_1^3 b_3^3 + b_1^3 b_2^3}$ which have the property
$\sum_{i=1}^{3} w_i = 1; w_i|_{e_j} = \delta_{ij}; \partial^T w_i|_{e_i} = 0; (r = 1, 2)$, it follows the Theorem on the next slide.

Side-Vertex-Method

Theorem: Geometric Surface Patch (Hagen-1986)

For a function from $C^3_T :=$ $\{F : F \in C'(T); \frac{\partial^{n+m}}{\partial x^n \partial y^m}|_{S_i} \in C^0(S_i); n+m=3; i=1,2,3\}$, with T as any triangle, interpolates

$$P[F] := \frac{b_2^2 b_3^2 P_1[F] + b_1^2 b_3^2 P_2[F] + b_1^2 b_2^2 P_3[F]}{b_2^2 b_3^2 + b_1^2 b_3^2 + b_1^2 b_2^2}$$

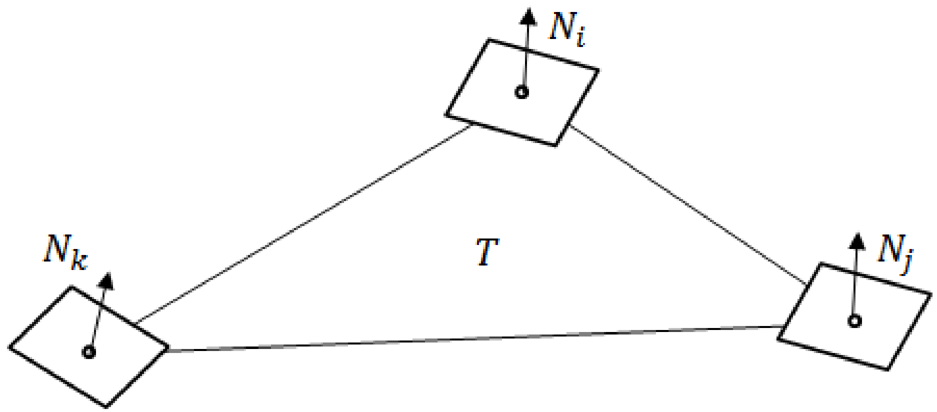
the tangential derivatives and the curvature vectors along the boundary ∂T.

Side-Vertex-Method

In this scheme, one has the possibility to use not only positions and tangential requirements but also curvture entries, since one can enter normal cutting curvature and geodetic curvaturealong the boundary and this data can be reproduced exactly.
If one has "information transverse to the edges" and these also are to be introduced in the modeling process, so it is difficult with the methods described thus far. This task is solved by the so called G^1- and G^2-surface patches, which will nor be presented.

Side-Vertex-Method

A G^1-surface patch (G^1-first order geometric continuity) has in each point a clearly tangential plane and the interpolation task is to interpolate the normal vectors along the boundary curves.



Side-Vertex-Method

This problem can be brief: It is looked for a interpolation scheme $P[F]$, which satisfies the following conditions:

$$P[F](\partial T) = F(\partial T)$$

$$N[P[F]](\partial T) = N[F](\partial T)$$

Side-Vertex-Method

The first essential task to solve the problem is the definition of the univariant operators

$$g[V_0, V_1, N_0, N_1](t) := H_0(t)V_0 + H_1(t)V_1$$

$$+\alpha\overline{H}_0(t)\frac{[[N_0, V_1 - V_0], N_0]}{||[[N_0, V_1 - V_0], N_0]||} + +\beta\overline{H}_1(t)\frac{[[N_1, V_1 - V_0], N_1]}{||[[N_1, V_1 - V_0], N_1]||}$$

($\{H_0, H_1, \overline{H}_0, \overline{H}_1\}$ is a cubic hermite basis and α, β have the role of tension parameters)

Side-Vertex-Method

This opertaor has the wished properties:

$$g(0) = V_0; g(1) = V; < g'(0), N_0 >= 0; < g'(1), N_1 >= 0$$

The g-operator is now linked with the radial operator:

$$G_i[F] := g[F(V_i, F(\frac{b_j V_j + b_k V_k}{1 - b_i}), N[F](V_i), N[F](\frac{b_j V_j + b_k V_k}{1 - b_i})](1-b_i)$$

Side-Vertex-Method

Theorem: G^1-surface-patch (Nielson-1987)

$$G[F] := W_1 G_1[F] + W_2 G_2[F] + W_3 G_3[F]$$

with $w_i = \frac{b_j^2 b_k^2}{b_1^2 b_2^2 + b_2^2 b_3^2 + b_1^2 b_2^2}$ satisfies the interpolation conditions $P[F](\partial T) = F(\partial T)$ and $N[P[F]](\partial T) = N[F](\partial T)$.

Side-Vertex-Method

To get now a G^2-surface-patch, a "geometric modification" of the quintic hermite operator is necessary.

$$H_5[V_0, V_1, V_0', V_1', V_0'', V_1''](t) =$$

$$H_0^5(t)(V_0)+H_1^5(t)(V_0')+H_2^5(t)(V_0'')+H_3^5(t)(V_1)+H_4^5(t)(V_1')+H_5^5(t)(V_1'')$$

$V_0' = \lambda_0 T_0; V_1' = \lambda_1 T_1;$
$V_0'' = \lambda_0^2 K_0 + \mu_0 T_0; V_1'' = \lambda_1^2 K_0 + \mu_1 T_1; \lambda_i > 0$

Side-Vertex-Method

We choose now the following "surface reference" for the curvature vectors K_i:

$$K_0 := K_0^N N_0; K_1 := K_1^N N_1$$

and define as an extension the Nielson-operator

$$\overline{g}[V_0, V_1, C_0, C_1](t) :=$$

$$H_5[V_0, V_1, \lambda_0 T_0, \lambda_1 T_1, \lambda_0^2 K_0 + \mu_0 T_0, \lambda_1^2 K_1 + \mu_1 T_1](t)$$

Side-Vertex-Method

This operator is now linked with the radial operator, where the abbreviate spelling "curvature-element" C^0 and C^1 each contain the normals.

$$P_i[F] := \overline{g}[F(V_i), F(S_i), C[F](V_i), C[F](S_i)](1 - b_i)$$

$$S_i = \frac{bjVj + b_k V_k}{1 - b_i}$$

Side-Vertex-Method

Theorem: G^2-Surface-Patch (Hagen-Pottmann-1988)

$$P[F] := W_1 P_1[F] + W_2 P_2[F] + W_3 P_3[F]$$

with $W_i := \frac{b_j^3 b_k^3}{b_1^3 b_3^3 + b_2^3 b_3^3 + b_3^3 b_1^3}$ satisfies the interpolation conditions $P[F](\partial T) = F(\partial T)$ and $C[P[F]](\partial T) = C[F](\partial T)$.

Discretization of the transfinite patches

A common form of the discretization of the space representations, based on the blending function method, is the cubic hermite interpolation of the positions at the vertices and the linear interpolation of the tangents and overtangents.
In this way one leads transfinite triangular patches in a "nine parametric" discrete form. The patch then only depends on the three position vectors of the triangle points and the respective two tangent vectors at the vertices.

Discretization of the transfinite patches

The case of G^1- and G^2-surface patches must be proceed different. We denote the vertices with F_1, F_2, F_3 and the corner normals with N_1, N_2, N_3.
The positions along the boundary is get by applying the Nielson-operators:

$$\begin{aligned} F(1-t, t, 0) &:= g[F_1, F_2, N_1, N_2](t) \\ F(0, 1-t, t) &:= g[F_2, F_3, N_2, N_3](t) \\ F(t, 0, 1-t) &:= g[F_3, F_1, N_3, N_1](t) \end{aligned}$$

Discretization of the transfinite patches

The normals can not be produced by linear interpolation of the corner normals. It is rather useful, to "normal install" the linear variation of the cross tangents.

$$N[F](1-t,t,0) = N_3(t) :=$$

$$\frac{[\frac{\partial F}{\partial e_3}, (1-t)[N_1, \frac{\partial F}{\partial e_3}(0)] + t[N_2, \frac{\partial F}{\partial e_3}(1)]]}{|| -'' - ||}$$

$$N[F](0,1-t,t) = N_1(t) :=$$

$$\frac{[\frac{\partial F}{\partial e_1}, (1-t)[N_2, \frac{\partial F}{\partial e_1}(0)] + t[N_3, \frac{\partial F}{\partial e_1}(1)]]}{|| -'' - ||}$$

$$N[F](t,0,1-t) = N_2(t) :=$$

$$\frac{[\frac{\partial F}{\partial e_2}, (1-t)[N_3, \frac{\partial F}{\partial e_2}(0)] + t[N_1, \frac{\partial F}{\partial e_2}(1)]]}{|| -'' - ||}$$

Discretization of the transfinite patches

With this normal representation we now can define the "discret" form.

$$g_i[F] := g[F_j, F_k, N_j, N_k]\left(\frac{b_j V_j + b_k V_k}{1 - b_i}\right), N_i, N_i\left(\frac{b_j V_j + b_k V_k}{1 - b_i}\right)(1 - b_i)$$

g_i interpolates now all(!) position inputs along the boundary and the surface normal along the edge e_i. Thus the convex combination schemes are getting easier than in the transfinit case.

Discretization of the transfinite patches

Theorem: Discrete-G^1-Surface-Patch (Nielson.1987)

$$A[F] := \frac{b_2 b_3 g_1[F] + b_1 b_3 g_2[F] + b_1 b_2 g_3[F]}{b_2 b_3 + b_1 b_3 + b_1 b_2}$$

interpolates the corner points and tangential planes in the corders of any triangle.

Discretization of the transfinite patches

In the case of the G^2-surface-patches one interpolates along the boundary threw:

$$\begin{aligned} F(1-t,t,0) &:= g[F_1,F_2,N_1,N_2](t) \\ F(0,1-t,t) &:= g[F_2,F_3,N_2,N_3](t) \\ F(t,0,1-t) &:= g[F_3,F_1,N_3,N_1](t) \end{aligned}$$

A linear variant of the cross tangent is here now not compatible with the curvature vectors C_i, thus

$$N_i(t) := \frac{[\frac{\partial F}{\partial e_i}(1-f_i(t))[N_j,\frac{\partial F}{\partial e_i}(0)]+(f_i(t))[N_k,\frac{\partial F}{\partial e_i}(1)]]}{\_\_ '' \_\_}$$

Discretization of the transfinite patches

In the corner points are the to the boundary tangent conjugated directions $R_i(t)$ given by the curvature elements C_i. One get $f_i'(0)$ and $f_i'(1)$ Out of the characterization of the conjugated directions. Because of $f_i'(0) = 0$ and $f_i'(1) = 1$ one gets $f_i(t)$ using a hermite interpolation and thus $N_i(t); i = 1, 2, 3$.
The normal curvature χ^N can linearly vary

$$\chi^N(R_k(t)) = (1 - t)\chi^N(R_k(0)) + t\chi^N(R_k(1))$$

Discretization of the transfinite patches

With this normal- and curvature representations we now give the diskrete form:

$$p_i[F] := g[F_i, g[F_j, F_k, C_j, C_k]$$

$$(\frac{bjVj + b_k V_k}{1 - b_i}), C_i, C_i(\frac{bjVj + b_k V_k}{1 - b_i})](1 - b_i)$$

g_i interpolates now all(!) position inputs along the boundary and the surface normal along the edge e_i. Thus the convex combination schemes are getting easier than in the transfinit case.

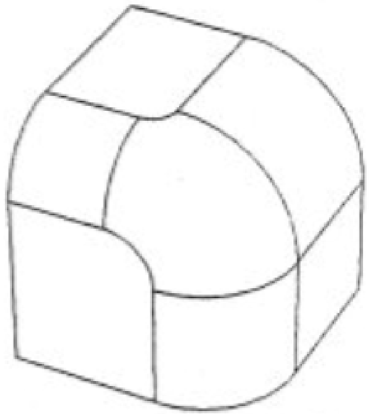
Discretization of the transfinite patches

Discrete-G^2-Surface-Patch (Hagen-Pottmann-1988)

$$p[F] := \beta_1 p_1[F] + \beta_2 p_2[F] + \beta_3 p_3[F]$$

with $\beta_i := \frac{b_j^2 b_k^2}{b_1^2 b_2^2 + b_1^2 b_3^2 + b_2^2 b_3^2}$ interpolates the corner points and the curvature elements in the corners of any triangle.

Vertex Blend Surfaces using N-side patches



$$X(u,w) := \sum_{i=0}^{n-1} \alpha_i(u,w) Q_i(u,w)$$

$$\sum_{i=0}^{n-1} \alpha_i = 1$$

Vertex Blend Surfaces using N-side patches

local parameter:

$$s_i(u,w) := \frac{1 - u \cdot \cos(2(i-1)\cdot\theta) - w \cdot \sin(2(i-1)\cdot\theta}{2 - 2\cos(s\theta)[u\cdot\cos(2i\theta) + w\cdot\sin(2i\theta)]}$$

$0 \leq s_i \leq 1; t_i := 1 - s_{i-1}$
These local parameter are singular for $n = 3$, use barycentric coordinates.

Vertex Blend Surfaces using N-side patches

local interpolate:

$$Q_i(s_i, t_i) := C_i(s_i) + C_{i-1}(s_{i-1}) - C_i(0) + t_i D_i^+(s_i) + s_i D_{i-1}^-(s_{i-1})$$

$$-t_i D_i^+(0) - s_i D_{i-1}^-(1) - s_i t_i T_i$$

D_i^+, D_i^- : cross tangents; T_i: twist vector

Example: $T_i = \frac{s_i D_i'^+ + t_i D_{i-1}'}{s_i + t_i}$

Vertex Blend Surfaces using N-side patches

blending function:

$$\alpha_i := \frac{\beta_i}{\sum_{j=0}^{n-1} \beta_j}$$

$\beta_i := \prod_{j \neq i,i+1} s_j^2$ or $\beta_i := \prod_{j \neq i,i+1} s_j t_{j-1}$ or $\beta_i := \prod_{j \neq i-1,i} d_j^2$
(d_j : eucl. distance)