Geometric Modelling Summer 2018

– Exercises –

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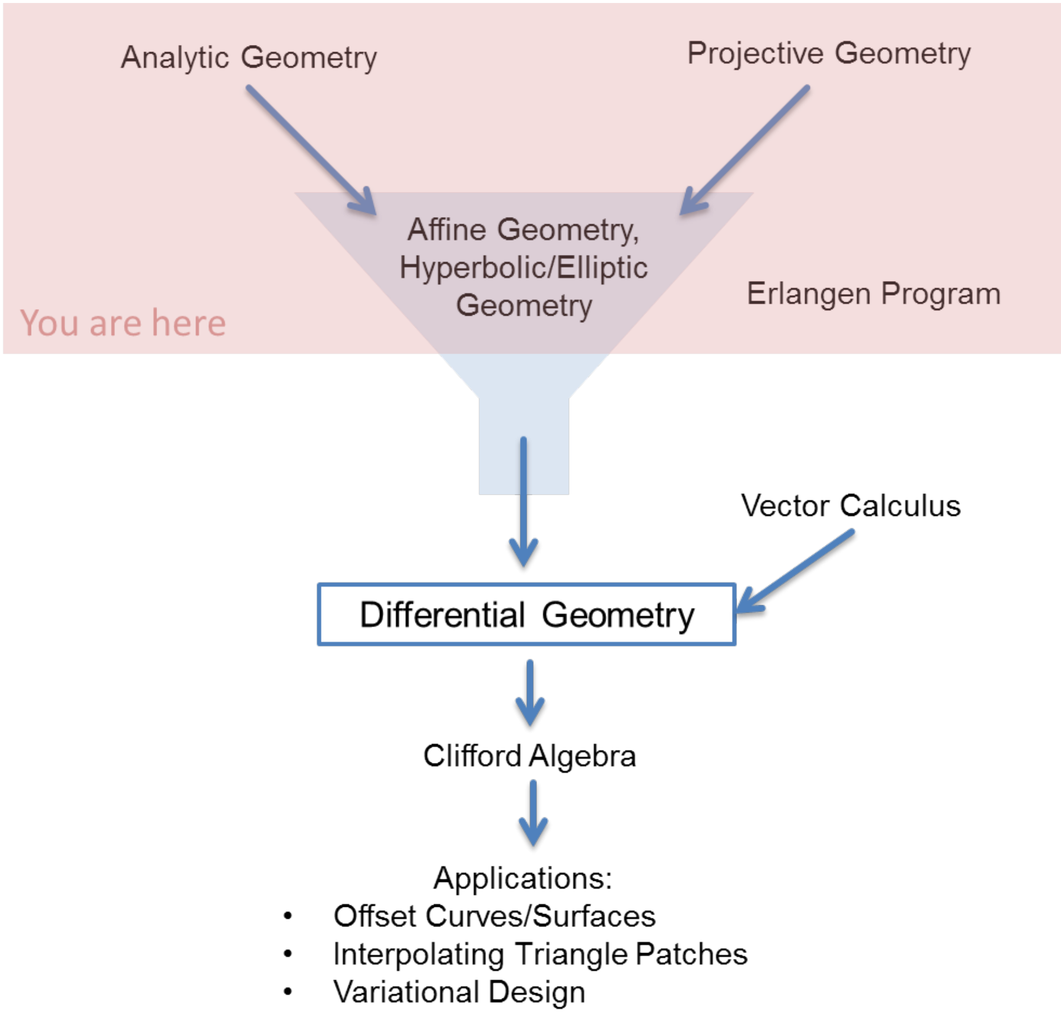
http://hci.uni-kl.de/teaching/geometric-modelling-ss2018

Analytic and Projective Geometry

Course Progess



Notes on Higher Order Vector Spaces

- In this course: exterior product mainly introduced for areas and volumes
- In 3d, this is done by the cross product ($\vec{a} \times \vec{b} = \vec{a} \wedge \vec{b}$ in 3d)
- Higher dimensions: Use Wedge-product to determine a subspace of proper dimension

Theorem: "Volume Property of the Determinant"

$\|\vec{a}_1 \wedge \vec{a}_2 \wedge \ldots \wedge \vec{a}_k\|$ is the volume of the k-dim. parallelotope spanned by $\vec{a}_1$, $\vec{a}_2$, ..., $\vec{a}_k$ in $\mathbb{E}^n$ $(k < n)$:

$$\|\vec{a}_1 \wedge \ldots \wedge \vec{a}_k\| = \begin{vmatrix} \langle \vec{a}_1, \vec{a}_1 \rangle & \ldots & \langle \vec{a}_1, \vec{a}k \rangle \\ \vdots & \ddots & \vdots \\ \langle \vec{a}_k, \vec{a}_1 \rangle & \ldots & \langle \vec{a}_k, \vec{a}_k \rangle \end{vmatrix}$$

Projective Geometry

Geometric Motivation

- Example: Tranfering objects from $\mathbb{R}^2$ into P^2
- An artist with eye O draws objects from an infinite floor F on an infinite canvas C
- Basically, this is done by by calculating the intersection of the line through O and a point P with C

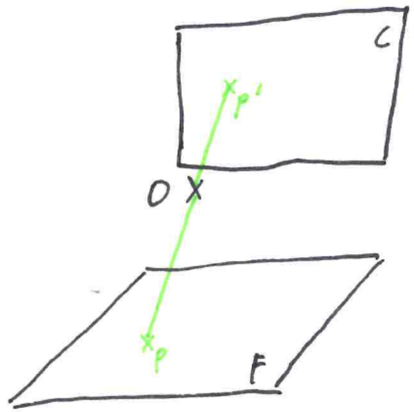


Figure: Setup for the projection from F to C

Projective Geometry

Geometric Motivation

- The artist cannot draw points whose connection to O would be parallel to $F \rightarrow$ Horizon
- Parallel lines in F meet in the horizon in C
- In real images, the Horizon usually bisects the image into the part in front of the artist and e.g. the sky

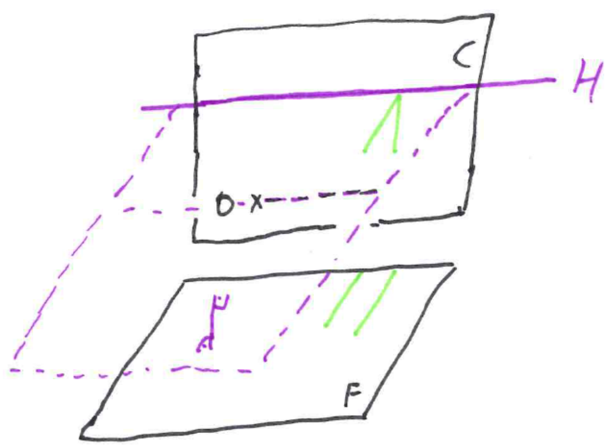


Figure: The horizon H. Points in this plane cannot be projected from F to C.

Projective Geometry

Geometric Motivation

- The border dinstincting "behind" and "in front of" the artist is marked by the plane parallel to C containing O
- Objects crossing this border are split into two, one half projected upside down
- Lines meeting in this border are parallel on C

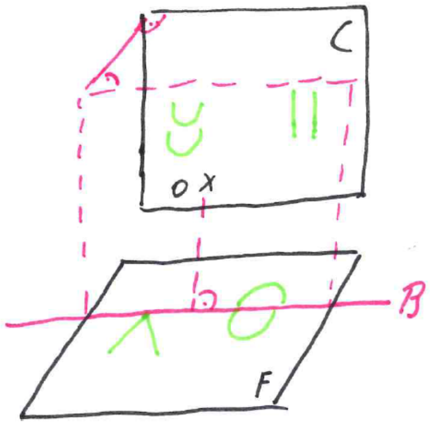


Figure: Border B. Sometimes, the projection may be restricted to one side of this border.

Projective Geometry

Exercise

Let the floor F be denoted by $z = 0$, the canvas C by $x = 0$ and the eye $O = (1, 1, 1)^T$.

a) Find the Horizon.

b) Find the image on C of a line parameterized in F by $ax + by = c$.

Projective Geometry

Solution
Let the floor F be denoted by $z = 0$, the canvas C by $x = 0$ and the eye $O = (1, 1, 1)^T$.

a) Find the Horizon.

Horizon: line in C defined hy $H = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + t \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

b) Find the image on C of a line parameterized in F by $ax + by = c$.

Projective Geometry

Solution

b) Find the image on C of a line parameterized in F by $ax + by = c$.
Procedure: 1) Solve for y: $y = \frac{1}{b}(c - ax) \to r$
2) compute the line from a point on that line through O in parametric two-point form:

$$r = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t \cdot \left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} x \\ \frac{c}{b} - x\frac{a}{b} \\ 0 \end{pmatrix} \right)$$

Projective Geometry

Solution

b) Find the image on C of a line parameterized in F by $ax + by = c$.
$r = (1,1,1)^T + t(1 - x, 1 - \frac{c}{b} + x\frac{a}{b}, 1)^T$
Procedure: 3) Intersect r with C:
Normal on C: $\vec{n} = (1,0,0)^T$. This normal is already normalized. Put r into the Hesse form of the plane to get the intersection:

$$\langle \vec{n}, r \rangle = 0 \Leftrightarrow \langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t \cdot \begin{pmatrix} 1 - x \\ 1 - \frac{c}{b} + x\frac{a}{b} \\ 1 \end{pmatrix} \rangle = 0$$

$$\Rightarrow 0 = 1 + t - xt$$

Projective Geometry

Solution

b) Find the image on C of a line parameterized in F by $ax + by = c$.
$r = (1,1,1)^T + t(1-x, 1-\frac{c}{b} + x\frac{a}{b}, 1)^T$ and $0 = 1 + t - xt$
Procedure: 4) Solve for t and insert result into r to get the intersection. Solving for t, we get $t = \frac{-1}{1-x}$, thus the intersection of $ax + by + c$ with C is given by:

$$P' = \begin{pmatrix}1\\1\\1\end{pmatrix} + \frac{-1}{1-x} \cdot \begin{pmatrix}1-x\\1-\frac{c}{b}+x\frac{a}{b}\\1\end{pmatrix}$$

$$= \begin{pmatrix}0\\1+\frac{1}{1-x}\cdot\left(\frac{c}{b}-x\frac{a}{b}-1\right)\\-\frac{x}{1-x}\end{pmatrix}$$

Projective Geometry

Computer Graphics

- In Computer Graphics, the scene is usually restricted to objects behind the canvas C
- The eye point (camera) is still in front of C
- Projection works the same as before $\rightarrow$ Pixel Shader

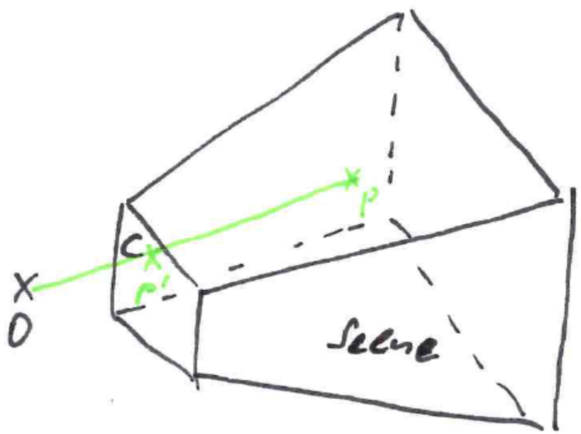


Figure: Setup for Pixel Shader. The cone limits the part of the scene rendered to the canvas C.

Projective Geometry

Computer Graphics

- Introduce homogeneous coordinates and make the (3d-)scene a projective space
- Then, the perspective transformation of the scene to the canvas is defined as follows:

Let d denote the distance of the eye to the canvas and the eye be at position $(0,0,d)^T$ looking into direction $-z$. The central projection of a point P in the scene to a point P' in the canvas is given by $P' = MP$, where

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -d^{-1} & 1 \end{pmatrix}$$

Projective Geometry

Principle of Duality
The principle of duality in projective spaces allows us to prove an "easier" theorem to get a proof for a less intuitive, more complicated phenomenon. When looking at a theorem in projective space, always keep in mind the consequences of the dual form.

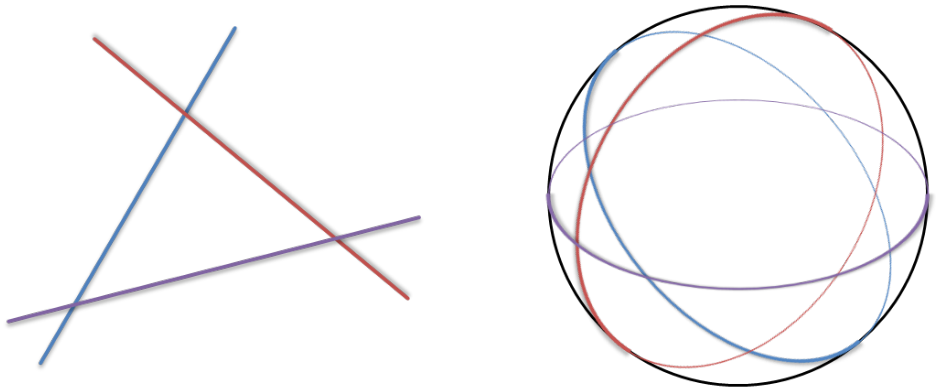
Exercise
Three pairwise non-parallel lines in $\mathbb{R}^2$ which do not all pass through the same point, disconnect the plane into seven connected components.

a) What happens if this is done in the projective plane?
b) Formulate the dual statement to the result you found.

Projective Geometry

Solution

a) Situation:



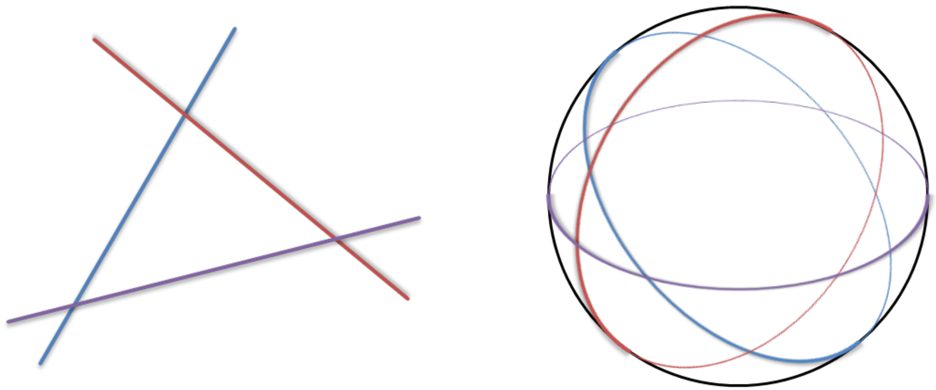
Observation: Three lines subdivide the projective plane into eight uniquely defined bordered subspaces.

b) Formulate the dual statement to the result you found.

Projective Geometry

Solution

b) Situation:



Dual Proposition: Three **points** in P^2 that are not **connected** by the same **line** define eight subspaces in P^2.

Note: As these subspaces are all triangles in P^2, it follows that three points do not uniquely determine a triangle in the projective plane!

Affine Geometry

Course Progess

Affine Transformations

Let $\vec{X}$ a set of points. The application of an operator or a mapping to $\vec{X}$ is interpreted as the apllication to every $\vec{x} \in \vec{X}$.
Remember from the lecture:

Affine Transformation

Affine Transformations are transformations that:

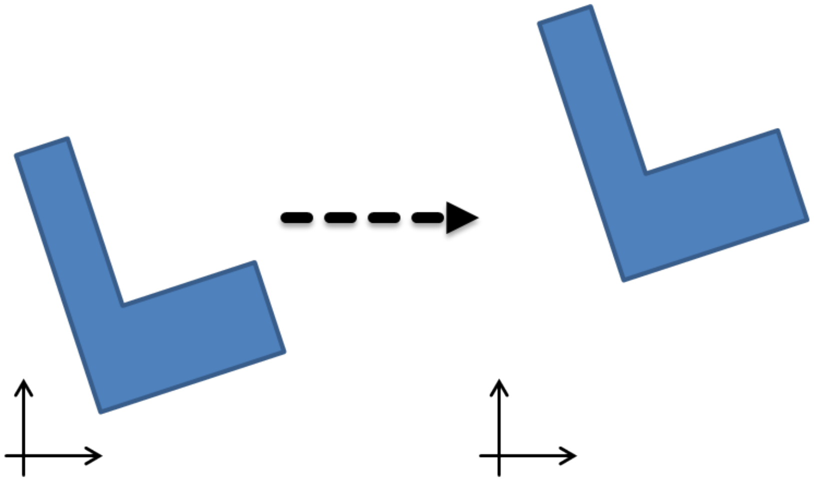
- preserve collinearity and coplanarity of points (i.e. preserve lines and planes)
- preserve ratios of vectors along lines
- preserve parallelity

They can be represented as "a linear transformation plus a vector":

$$\vec{y} = A\vec{x} + \vec{a} \quad \text{where } det(A) \neq 0$$

Affine Transformations

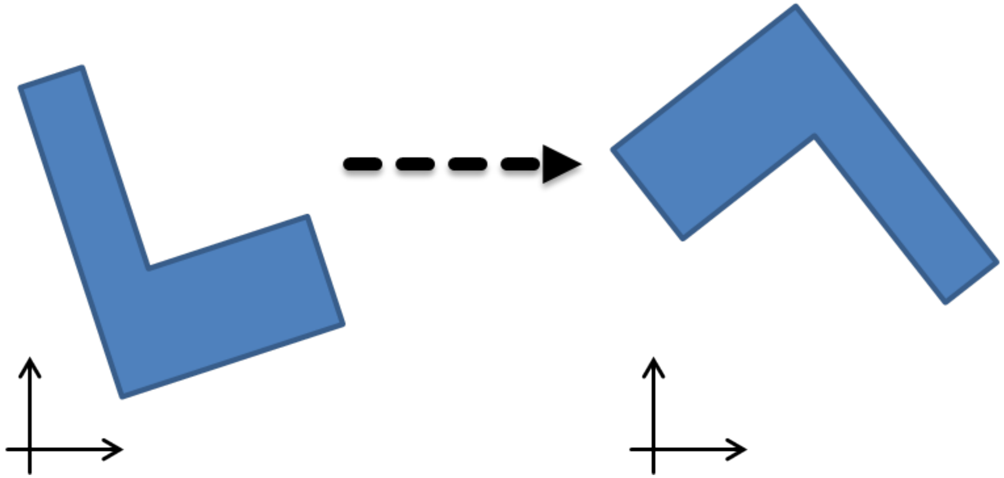
Translation: $\vec{Y} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vec{X} + \begin{pmatrix} t_x \\ t_y \end{pmatrix}$



Translation is a special case where the linear map is the identity.

Affine Transformations

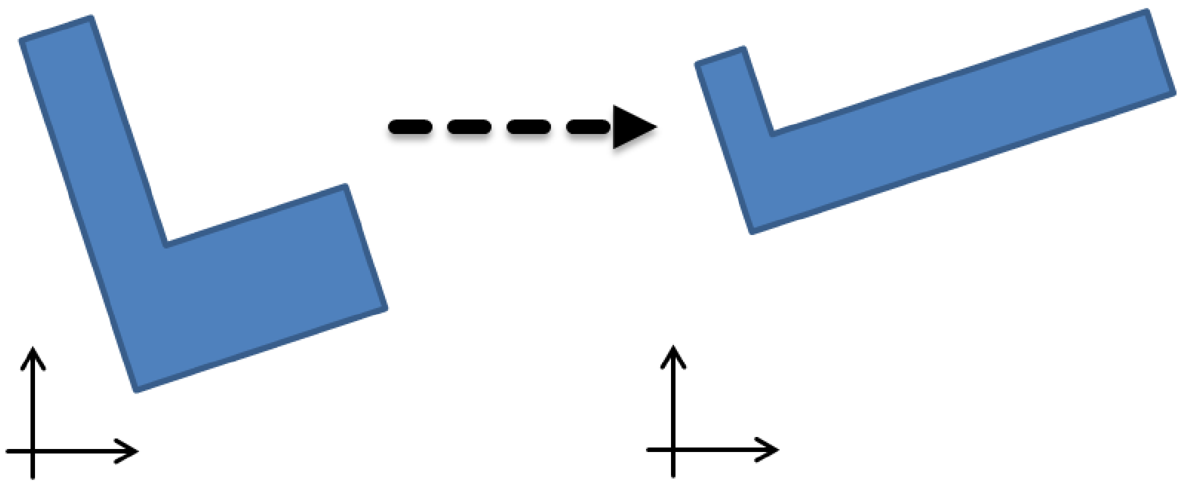
Rotation: $\vec{Y} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \vec{X} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}$



This linear equation provides the rotation around an angle φ for an object that is centered in the offspring. Generally, the object has to be translated to the offspring before rotation and moved back thereafter. To rotate around an arbitrary point, the offspring has to be moved into that point by translation of the coordinate system.

Affine Transformations

Scaling: $\vec{Y} = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \vec{X} + \begin{pmatrix} 0 \\ 0 \end{pmatrix}$



Scaling can destroy orthonormality, orthogonality, Cartesian system, right-hand-system (negative factors!), normal lengths, and normals!

Affine Transformations

For technical reasons, we would like to model all these transformations as linear maps. This would allow us to exploit the linearity of the maps and multiply the transformation matrices to encode multiple transformations in a single matrix.

Idea: use affine coordinates!

Affine Transformations

Remember from the lecture:

Integration of the Classical Geometries in the Sense of the Erlangen Program:
Consider the subgroup of the projective group (i.e. the corresponding geometry) which fixes a hyper plane. We so to say "tag" this **hyperplane as absolute figure at infinity** and restrict the effect of the subgroup of the transformation group to the points that are not incident with this hyperplane.
This precedure yields affine transformations by restricting the projective transformations to the points that are not at infinity.
The **affine group** then reveals itself as a **subgroup of the projective group**.

Affine Transformations

"The Procedure in Coordinates": P^3 ($x_0 = 0$ is mapped to $y = 0$)

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} a_{00} & 0 & 0 & 0 \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\rightarrow \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} a_{10} \cdot x_0 & a_{11} \cdot x_1 & a_{12} \cdot x_2 & a_{13} \cdot x_3 \\ a_{20} \cdot x_0 & a_{21} \cdot x_1 & a_{22} \cdot x_2 & a_{23} \cdot x_3 \\ a_{30} \cdot x_0 & a_{31} \cdot x_1 & a_{32} \cdot x_2 & a_{33} \cdot x_3 \end{pmatrix}$$

$\rightarrow$ restrict to the points not at infinity (divide by x_0)

$$\rightarrow \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} \frac{x_1}{x_0} & \frac{x_2}{x_0} & \frac{x_3}{x_0} \end{pmatrix}^T + \begin{pmatrix} a_{10} \\ a_{20} \\ a_{30} \end{pmatrix}$$

affine transformation

Affine Transformations

This means that e.g. P^2 can be constructed from the affine plane by adding a line at infinity whose points are the equivalence classes of parallel lines in the affine plane. Conversely, the affine group can be obtained from P^2 by removing an arbitrary line and all the points on it.

Applying this precedure backwards, e.g. for the rotation around the z-axis, we get:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}^T + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Affine Transformations

Add a line at infinity, e.g. $\begin{pmatrix}0 & 0 & 0 & x_0\end{pmatrix}^T$:

$$\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix} = \begin{pmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{pmatrix} \cdot \begin{pmatrix}\frac{x_1}{x_0} & \frac{x_2}{x_0} & \frac{x_3}{x_0}\end{pmatrix}^T + \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix} = \begin{pmatrix}0\cdot x_0 & \cos\varphi\cdot x_1 & -\sin\varphi\cdot x_2 & 0\cdot x_3\\ 0\cdot x_0 & \sin\varphi\cdot x_1 & \cos\varphi\cdot x_2 & 0\cdot x_3\\ 0\cdot x_0 & 0\cdot x_1 & 0\cdot x_2 & 1\cdot x_3\end{pmatrix}$$

Affine Transformations

Obtain the homogeneous representation, set $x_0 = 1$:

$$\begin{bmatrix} 1 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi & 0 \\ 0 & \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Affine Transformations

The homogeneous coordinate is often written as the last coordinate. The equation then becomes:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 & 0 \\ \sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{bmatrix}$$

Affine Transformations

For the other rotations and scaling, one obtains:

- rotation (x-axis): $$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi & 0 \\ 0 & \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{bmatrix}$$
- rotation (y-axis): $$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{bmatrix}$$
- scaling: $$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ 1 \end{bmatrix} = \begin{bmatrix} s_1 & 0 & 0 & 0 \\ 0 & s_2 & 0 & 0 \\ 0 & 0 & s_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{bmatrix}$$

Affine Transformations

Exercise

What is the map for the translation?

Affine Transformations

Solution
What is the map for the translation?

Analoguously to obove, one obtains:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_1 \\ 0 & 1 & 0 & t_2 \\ 0 & 0 & 1 & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{bmatrix}$$

Affine Transformations

Now, we can exploit the linearity of the maps to construct more complicated maps by multiplying these basic ones. For example, the rotation around an arbitrary axis can be reduced to the rotation around one of the coordinate axes by properly transforming the coordinate system first.

Exercise
The transformation matrices for rotations given above rotate objects around an axis with respect to the offspring. To rotate an object around one of its own axes, one would need to rotate with respect to the center of mass of this object. In complex scenes, this is usually not the offspring.
How can this problem be reduced to rotating with respect to the offspring?

Affine Transformations

Solution
How can the problem be reduced to rotating with respect to the offspring?
Transformation: The coordinate system has to be moved s.t. the offspring is in the point used for the rotation. After the rotation, the coordinate system has to be translated back to its original position. Let P the point that is the center of the rotation, $R_z(\varphi)$ the rotation itself. Then, by multiplying the transformation matrices, we get:

$$\mathcal{R}_z(P, \varphi) = T_P \cdot R_z(\varphi) \cdot T_P^{-1}$$

where T_P is the map that translates a given point by the position vector of P.