Geometric Modelling Summer 2018

– Exercises –

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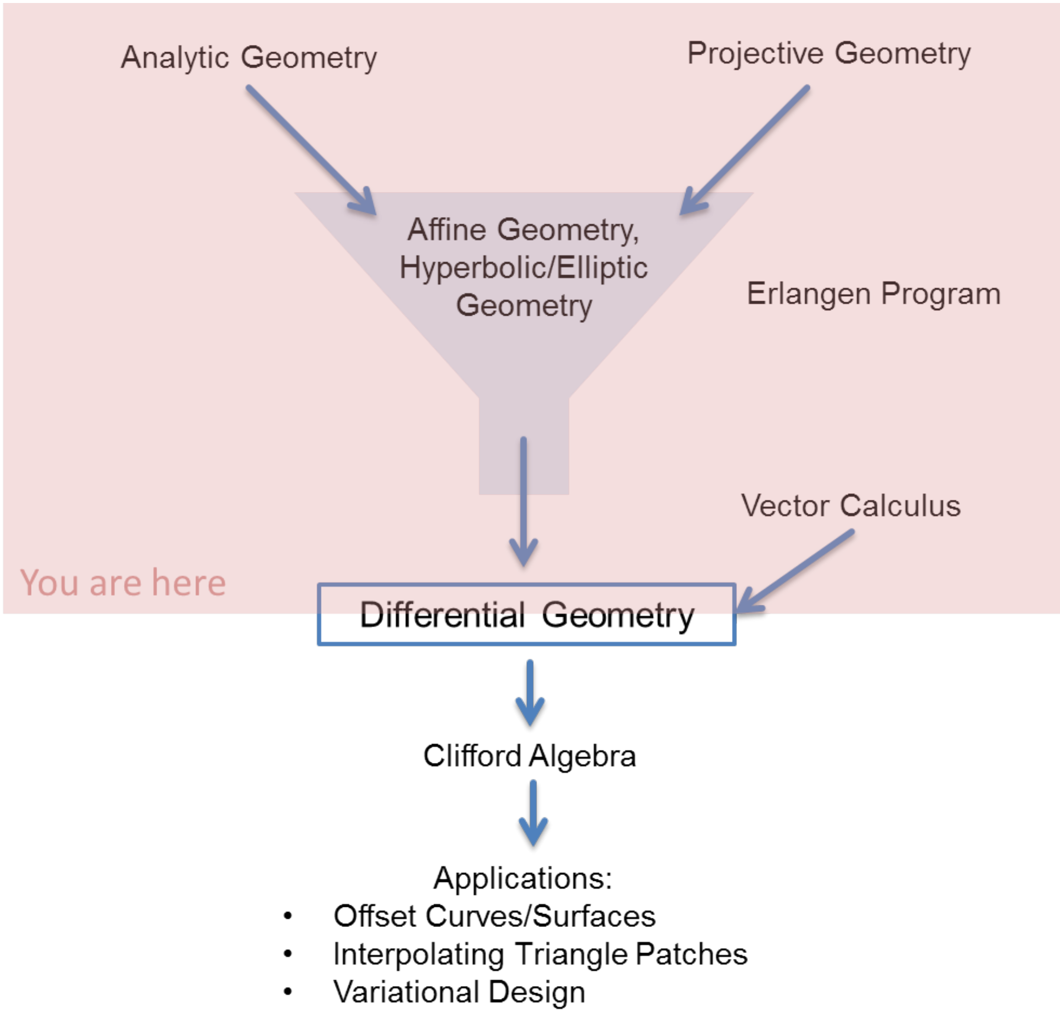
http://hci.uni-kl.de/teaching/geometric-modelling-ss2018

Differential Geometry – Curve Theory

Course Progess



Equivalence of Curves

- If the image of a curve is given, different parameterizations can be defined
- Differential Geometry: Describe properties of curves invariant under certain reparameterizations:
  $\rightarrow$ need to define a proper equivalence relation on parametric curves
- Properties of the equivalence class are invariant under reparameterization:
  - Length
  - Frenet Frame
  - Curvature, Torsion

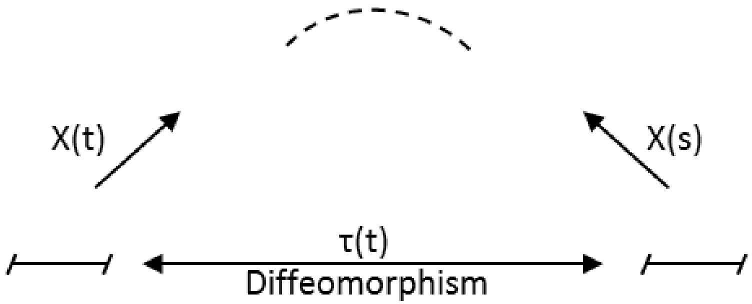
  These properties uniquely define a curve (up to Euclidean motion)

Equivalence of Curves

Equivalence of Curves, Reparameterization, Parameter Transformation

Two curves $X : M \to \mathbb{R}^n$ and $Y : N \to \mathbb{R}^n$ are **equivalent** if there exists a bijective C^r-map $\phi : M \to N$ s.t. for $t \in M$ $\phi'(t) \neq 0$ and $Y(\phi(t)) = X(t)$.
Y is called the **reparameterization** of X, ϕ is called a **parameter transformation** for X.

Note that this is equivalent to the definition given in the lecture.



„Reparametrization“

Arc Length Parameterization

The Length of a Curve Segment:

Define the length of a $\mathcal{C}^1$ curve $X : t \rightarrow \mathbb{R}^n$:

$$L(a, b) = \int_a^b \|X'(t)\| dt$$

This length is invariant under reparameterization and thus a differential geometric property.

If the curve is given as a parametric curve in Cartesian coordinates $X(t) = (x_1(t), \ldots, x_n(t))^T$, the equation becomes:

$$L(\alpha, \beta) = \int_\alpha^\beta \sqrt{\dot{x_1}^2 + \ldots + \dot{x_n}^2} dt$$

where α and β are the values of t at $x = a$ and $x = b$ respectively.

Arc Length Parameterization

Computation of the Arc Length Parameterization:
For a regular $\mathcal{C}^r$-curve (i.e. $r \geq 1$), we can define the length of the arc from 0 to t:

$$s(t) = \int_0^t \|X'(t)\| dt$$

Now, we need the inverse of this curve, $t(s)$ and reparameterize the curve by replacing every occurence of the parameter t by $t(s)$.

Again, if the curve is given as a parametric curve in Cartesian coordinates $X(t) = (x_1(t), \ldots, x_n(t))^T$, the equation becomes:

$$s(t) = \int_0^t \sqrt{\dot{x_1}^2 + \ldots + \dot{x_n}^2} dt$$

Reparameterization yields: $X(t(s) = (x_1(t(s)), \ldots x_n(t(s)))^T$.

Arc Length Parameterization

Example: The circle $X(t) = (r\cos t, r\sin t, 0)^T$ is to be parameterized by the arc length.

First, calculate the length of the tangent vector from 0 to t:

$$s(t) = \int_0^t \|X'(t)\| dt = \int_0^t \sqrt{(-r\sin t)^2 + (r\cos t)^2 + 0^2} dt$$
$$= \int_0^t \sqrt{r^2 \cdot (\sin^2 t + \cos^2 t)} dt = \int_0^t r dt = rt$$

Now, the inverse function of $s(t) = rt$ is $t(s) = \frac{s}{r}$.
Reparameterizing the original circle by replacing t by $\frac{s}{r}$, we obtain the arc length paramaterization:

$$X(s) = \left(r\cos\frac{s}{r}, r\sin\frac{s}{r}, 0\right)^T$$

Arc Length Parameterization

Remarks:

- Curves are only distinguished by the ways they bend (curvature) and twist (torsion)
- Arc length paramaterization helps for theoretical arguments as for a particle moving along the curve, all curves would "look the same" without further information about the space
- In practice often difficult to calculate (helpful: piecewise linear approximations)
- For a given parameterized curve X, the arc length parameterization is unique up to parameter shift

Arc Length Parameterization

Remarks:

- Interesting for technical applications: In arc length parameterization, a particle s traverses the curve X at unit speed ($\|X'(s)\| = 1 \ \forall s$)
- If the arc length parameterization is known, it is easy to calculate:
  - The Frenet frame: with $\|\dot{X}\| = 1$ and $\|\ddot{X}\| \neq 0$:
    - Tangent vector: $T = \dot{X}$ (this is a unit vector as $\|\dot{X}\| = 1$)
    - Principal normal vector: $N = \frac{\ddot{X}}{\|\ddot{X}\|}$
    - Binormal vector: $B = [T, N]$
  - The curvature: $\kappa = \|\dot{T}\| = \|\ddot{X}\|$
  - The torsion: $\tau = \langle -N, \dot{B} \rangle$

Arc Length Parameterization

Exercise

Provide a parameterization by the arc length for the following parameterized curve:

$$x\colon [0,1] \to \mathbb{R}^3 \text{ with } x(t) = \left(e^t \cos(t), e^t \sin(t), e^t\right)^T \text{ where } t \in [0,1]$$

Arc Length Parameterization

Solution

First, the tangent vector:
$\dot{x} = (e^t \cos t - e^t \sin t, e^t \sin t + e^t \cos t, e^t)^T$.
Now we compute the arc length:

$$\int_0^t \|\dot{x}\| dt = \int_0^t \sqrt{\dot{x}} dt$$
$$= \int_0^t \sqrt{2e^{2t}(\cos^2 t \sin^2 t) + e^{2t}} dt$$
$$= \int_0^t \sqrt{3}e^t dt = \sqrt{3}e^t \Big|_0^t = \sqrt{3}(e^t - 1)$$

So, the arc length is $s(t) = \sqrt{3}(e^t - 1)$ for $t \in [0, 1]$.

Arc Length Parameterization

Solution

...

So, the arc length is $s(t) = \sqrt{3}(e^t - 1)$ for $t \in [0, 1]$.
We now compute the borders $s(0) = 0$ and $s(1) = \sqrt{3}(e - 1)$ for the inverse of $s(t)$:

$$s(t) = \sqrt{3}(e^t - 1) \Leftrightarrow t = \ln\left(\frac{s(t)}{\sqrt{3}} + 1\right) \xrightarrow{\text{invert}} t(s) = \ln\left(\frac{s}{\sqrt{3}} + 1\right)$$

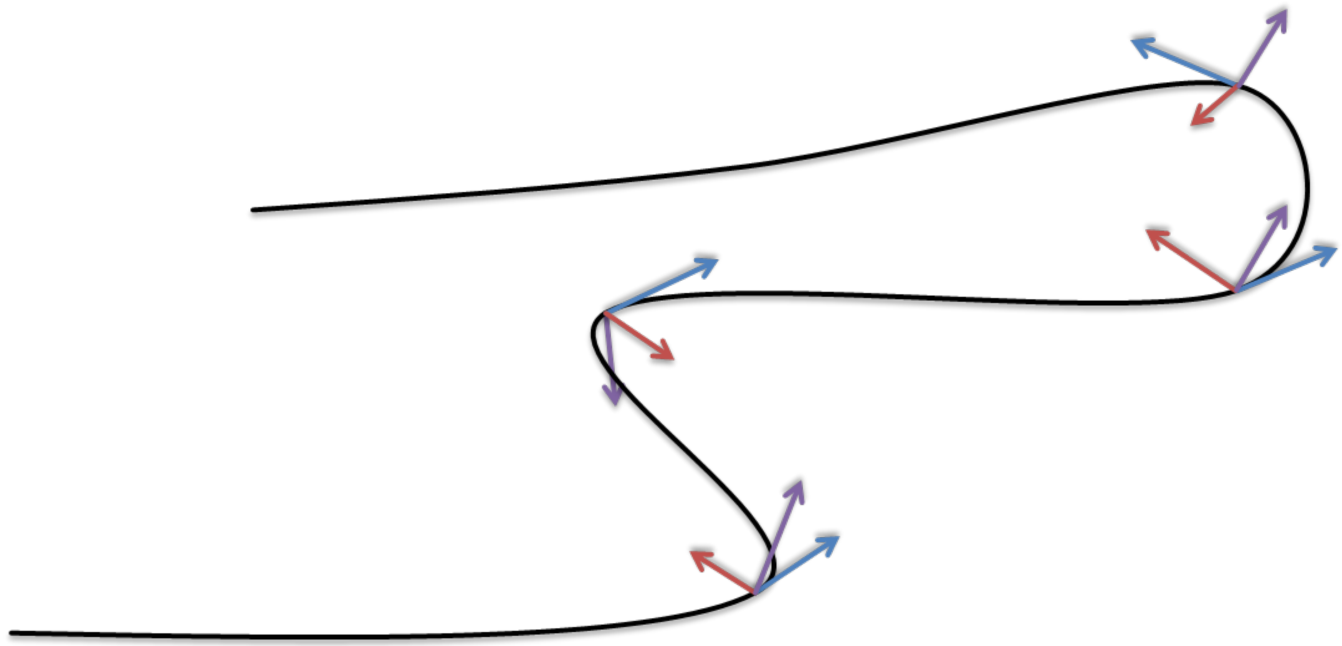
Reparameterizing the original representation, we obtain:

$$x(s) = \left(\frac{s}{\sqrt{3}} + 1\right) \cdot \begin{pmatrix} \cos\ln\left(\frac{s}{\sqrt{3}} + 1\right) \\ \sin\ln\left(\frac{s}{\sqrt{3}} + 1\right) \\ 1 \end{pmatrix} \qquad 0 \le s \le \sqrt{3}(e - 1)$$

Frenet Frame, Curvature, and Torsion

Frenet Frame:

- Tangent vector T (blue): direction of the curve
- Principal normal vector N (red): points to the center of the osculating circle
- Binormal vector B (purple): $[T, N]$



Frenet Frame, Curvature, and Torsion

Frenet Equations, Curvature and Torsion:

If a curve parameterized by the arc length is given, we can compute the vectors of the frenet frame, the curvature and the torsion as follows:

- The Frenet frame: with $\|\dot{X}\| = 1$ and $\|\ddot{X}\| \neq 0$:
  - Tangent vector: $T = \dot{X}$
  - Principal normal vector: $N = \frac{\ddot{X}}{\|\ddot{X}\|}$
  - Binormal vector: $B = [T, N]$
- The curvature: $\kappa = \|\dot{T}\| = \|\ddot{X}\|$
- The torsion: $\tau = \langle -N, \dot{B} \rangle$

More general equations are provided in the lecture slides.

Note that the principal normal vector and the binormal vector are not defined for curves that are not at least $\mathcal{C}^2$! For curves that are not $\mathcal{C}^2$, the torion is 0, i.e. they are planar.

Frenet Frame, Curvature, and Torsion

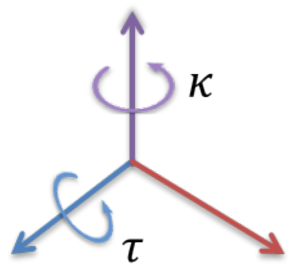
Intuition Behind Curvature and Torsion:
The curvature and torsion answer questions about the bending and the twisting behavior of a curve:

- curvature: How much does the curve deviate from a straight line?
- torsion: How much does the curve deviate from planar shape?

If the curvature is 0, the curve is a straight line. If the torsion is 0, the curve can be embedded in a plane.

Frenet Frame, Curvature, and Torsion

Connection Between Frenet Frame, Curvature and Torsion:
When the Frenet frame is moved along the curve, the curvature is the frame's rotation around the binormal. The torsion is the rotation around the tangent.



For example in a helix, the principal normal always points to the central axis (1) and is always orthogonal to it (2). Thus, if the frame moves along the helix, it has to rotate around the binormal to fulfill (1) and around the tangent to fulfill (2), thereby showing curvature and torsion.

Frenet Frame, Curvature, and Torsion

Exercise

Determine all arc length parameterized C^3-curves in $\mathbb{E}^3$ with constant curvature and torsion.

Frenet Frame, Curvature, and Torsion

Solution
Determine all arc length parameterized C^3-curves in $\mathbb{E}^3$ with constant curvature and torsion.

	$\kappa = 0$	$\kappa = \text{const} > 0$
$\tau = 0$	lines	circles
$\tau = \text{const} > 0$	–	helices

There is no object with vanishing curvature and nonzero torsion.
In fact, whenever the curvature vanishes, the torsion also does!
Question: Why?

Frenet Frame, Curvature, and Torsion

Solution

Whenever the curvature vanishes, the torsion also does!

Why?

→ Recall the formulas:

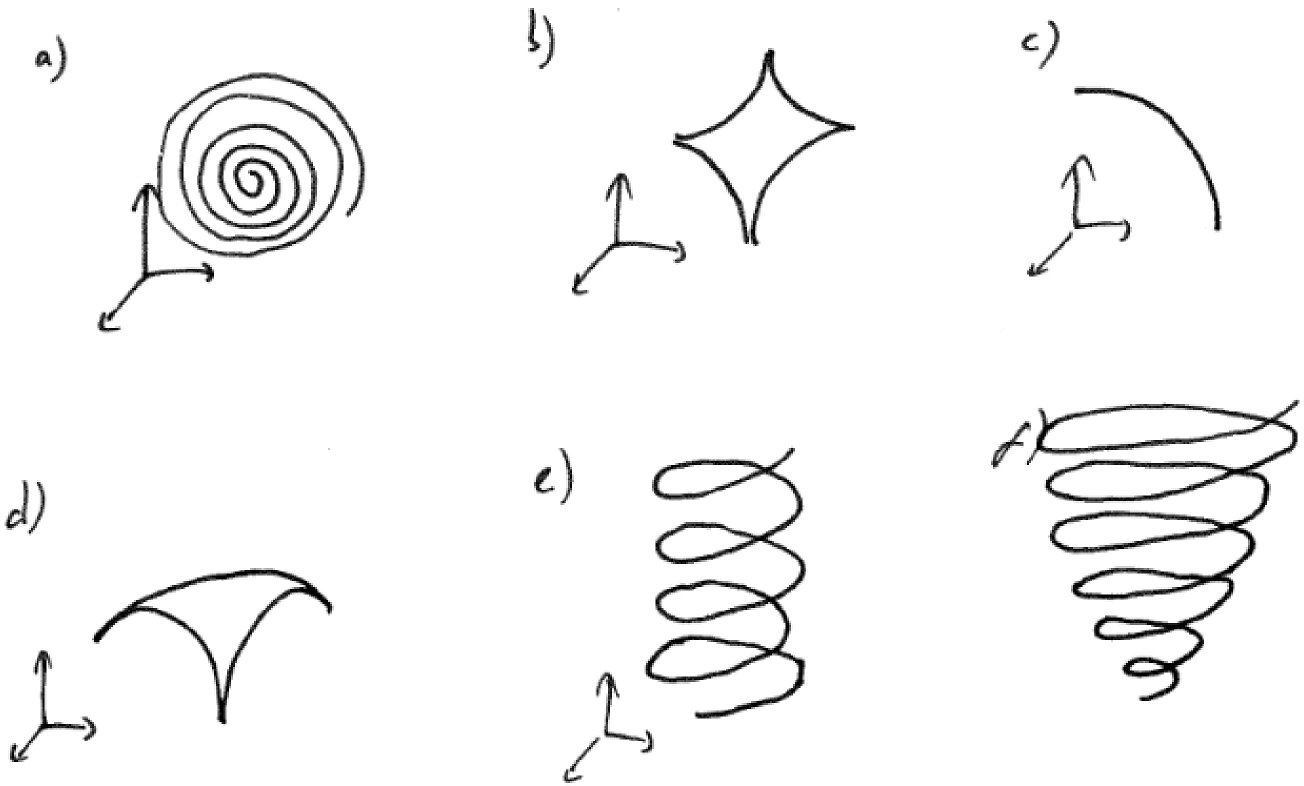
$$\begin{aligned}\kappa &= \| \dot{T} \| \\ \tau &= \langle -N, \dot{B} \rangle \\ &= \langle -\frac{\ddot{X}}{\|\ddot{X}\|}, \frac{d[T,N]}{dt} \rangle \\ &= \langle -\frac{\dot{T}}{\|\dot{T}\|}, \frac{d[T,N]}{dt} \rangle\end{aligned}$$

If the curvature vanishes, $\| \dot{T} \| = 0$. Therefore, T has to be a 0-vector and the torsion also vanishes.

Frenet Frame, Curvature, and Torsion

Exercise

For the following curves, decide for the curvature and the torsion whether they vanish, are constant, or vary:



To Establish

Solution

a) variable curvature, zero torsion
b) piecewise constant (but sign flipping) curvature, zero torsion
c) constant nonzero curvature, zero torsion
d) variable curvature and variable torsion
e) constant nonzero curvature and constant nonzero torsion
f) variable curvature, constant nonzero torsion