Geometric Modelling Summer 2018

– Exercises –

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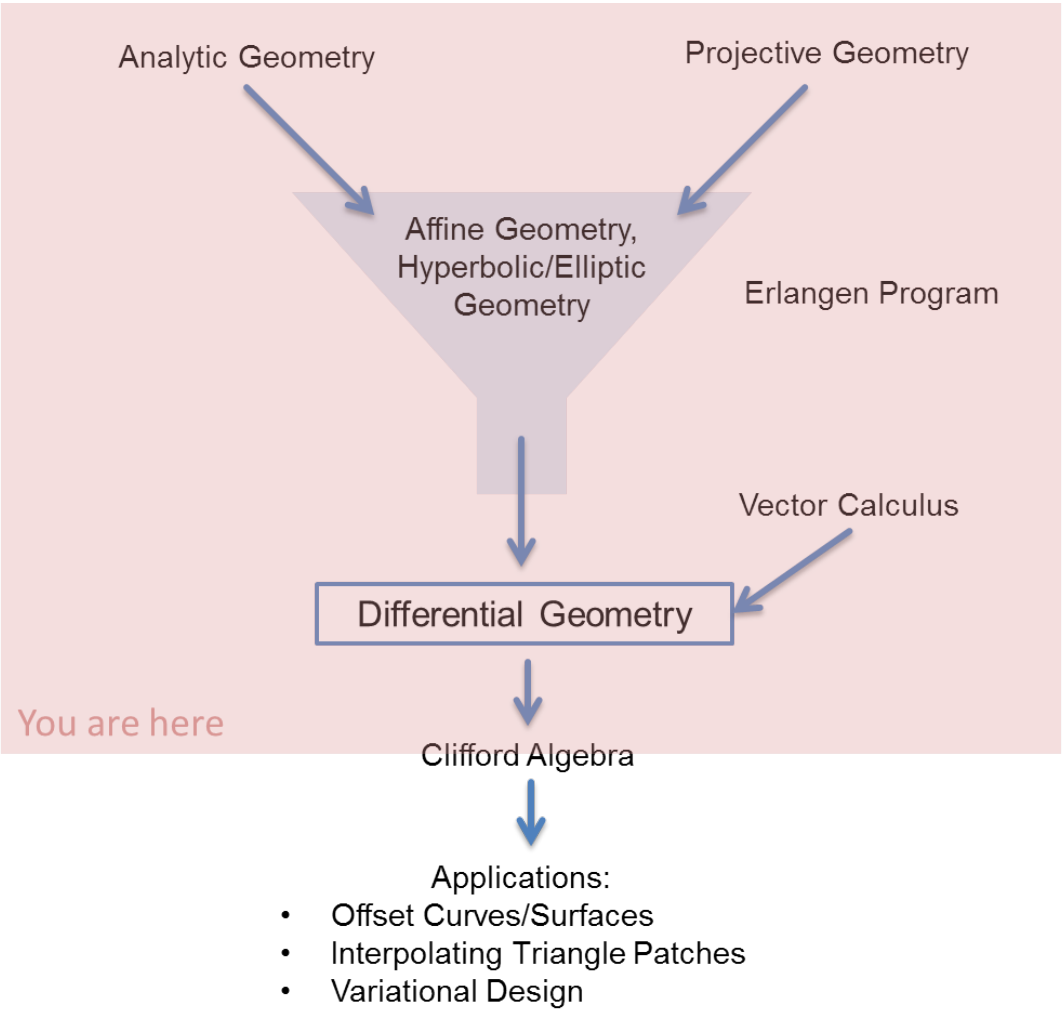
http://hci.uni-kl.de/teaching/geometric-modelling-ss2018

Differential Geometry – Surface Theory and Manifolds

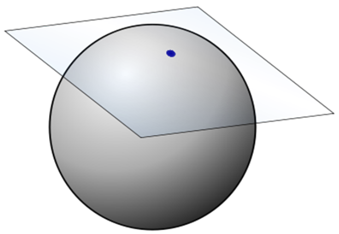
Course Progess

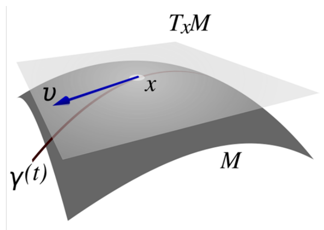


First Fundamental Form

Tangent Space

- generalization of vectors from affine spaces to general manifolds
- *intuition*: The space of all possible directions of a tangent through a point x on the surface
- *example*: Sphere: for every point, the plane perpendicular to the radius through x
- tangent spaces $\rightarrow$ define a vector field that smoothly assigns to every point x a vector from x's tangent space
- *analogue*: think of these vectors as velocities of a particle along a curve on the manifold

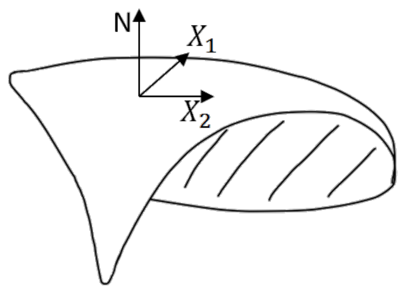




First Fundamental Form

Tangent Space

- special case: surface $X = X(u, v) \rightarrow X_u = \frac{\partial X}{\partial u}$, $X_v = \frac{\partial X}{\partial v}$
- in general: tangent space in point $X(u, v)$ is the plane spanned by X_u and X_v (exceptions: peak, ridge, ...)
- Gauss frame: analogue to Frenet frame: $\{X_u, X_v, N\}$ where unit normal vector $N = \frac{[X_u, X_v]}{\|[X_u, X_v]\|}$
- note: X_u and X_v do *not* need to be orthogonal, thus the Gauss frame is in general *not orthonormal*!



First Fundamental Form

First Fundamental Form

- inner product on tangent space, induced from scalar product
- completely describes the metric properties of a surface → can be used to compute lengths and areas on the surface

Definition: First Fundamental Form

Let $X(u,v)$ be a parametric surface. Then, the inner product of two tangent vectors in tangent space is given by:

$$I(x,y) = \langle x,y \rangle$$

$$I(x,y) = x^T \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} y$$

where $g_{ij} = \langle X_i, X_j \rangle$ for $i,j \in \langle 1,2 \rangle$, corresponding to the first resp. second parameter u, resp. v. Note: $g_{12} = g_{21}$.

First Fundamental Form

Exercise:
Compute I for the unit sphere.

First Fundamental Form

Exercise:
Compute I for the unit sphere.

Solution: Unit sphere: $X(u,v) = \begin{pmatrix} \cos u \sin v \\ \sin u \sin v \\ \cos v \end{pmatrix}$, where

$(u,v) \in [0, 2\pi) = \times [0, \pi)$.

$X_u = \begin{pmatrix} -\sin u \sin v \\ \cos u \sin v \\ 0 \end{pmatrix}$, and $X_v = \begin{pmatrix} \cos u \cos v \\ \sin u \cos v \\ -sinv \end{pmatrix}$

Coeffeicients $g_{ij} = \langle X_i, X_j \rangle$ for $i, j \in \langle 1, 2 \rangle$:

$g_{11} = sin^2 v$, $g_{12} = g_{21} = 0$, and $g_{22} = 1$

First Fundamental Form

First Fundamental Form
Line element:

$$ds^2 = g_{11} du^2 - 2g_{12} du\, dv + g_{22} dv^2$$

Surface element: Using Lagrange's identity
$\|a\|^2 \cdot \|b\|^2 - \langle a, b\rangle^2 = \sum_{1 \leq i < j \leq n (a_i b_j - a_j b_i)^2}$, one obtains:

$$dA = \|[X_u, X_v]\| du\, dv = \sqrt{g_{11} \cdot g_{22} - g_{12}^2} du\, dv$$

First Fundamental Form

Exercise:
Compute the length of the equator and the area of the overall surface of the sphere.

Solution: *Equator:* coefficients of I: $g_{11} = sin^2 v$, $g_{12} = g_{21} = 0$, and $g_{22} = 1$
Unit Sphere Equator $\rightarrow dv = 0$.

$$ds^2 = \sqrt{\sin^2 v \ du^2}$$

$$s = \int_0^{2\pi} \sin v du = 2\pi$$

First Fundamental Form

Exercise:
Compute the length of the equator and the area of the overall surface of the sphere.

Solution: *Surface Area:* coefficients of I: $g_{11} = sin^2 v$, $g_{12} = g_{21} = 0$, and $g_{22} = 1$

$$dA = \|[X_u, X_v]\| du\, dv = \sqrt{g_{11} \cdot g_{22} - g_{12}^2} du\, dv$$

$$= \sqrt{\sin^2 v \cdot 1 - 0} \text{v } du$$

$$A = \int_0^{2\pi} \int_0^{\pi} \sin v \text{v! u} = \int_0^{2\pi} 2\pi \sin v \text{v} = 4\pi$$

Note: This is the omputation for the unit sphere. For the general sphere, we obtain $4\pi r^2$ for radius r

Second Fundamental Form

Curvatures:

- remember from last time: for a particle moving along a curve ignorant of the "surroundings", all curves look the same $\rightarrow$ curves do not have a curvature unless embedded in some space $\rightarrow$ curvature is an extrinsic property for curves
- for planes, this is not true. They can have intrinsic curvature, *independent of an embedding* $\rightarrow$ Gaussian curvature
- In the following: repetition of concepts from the lecture

Second Fundamental Form

Curvatures: Curves on Surfaces:

- setting: 1d curve C on 2d surface X embedded in $\mathbb{R}^3$
- if the curve is non-singular its tangent vector lies in the plane's tangent space
- **normal curvature**: k_n curvature of the curve projected onto the plane spanned by $\dot{C}$ and surface normal vector N
- **geodesic curvature:** k_g curvature of the curve projected onto the surface's tangent plane
- **geodesic torsion:** τ_r measures the rate of change of the surface normal N around the curve's tangent $\dot{C}$

...

Second Fundamental Form

Curvatures: Curves on Surfaces:

...

- all curves with same tangent vector have the same normal curvature = the curvature obtained by intersecting the surface with the plane spanned by $\dot{C}$ and N
- **principal curvatures:** k_1 and k_2 are the maximum and minum values of the normal curvature at a point
- **principal directions:** the directions of the tangent vectors corresponding to k_1 and k_2
- this is equivalent to the definition via an eigenproblem given in the lecture

Second Fundamental Form

Curvatures: Gaussian Curvature:

$$K = k_1 \cdot k_2$$

- positive for spheres, negative for hyperboloids, zero for planes
- determines whether a surface is locally convex (= locally spherical) or locally saddle (= locally hyperbolic)
- this definition is extrinsic as it uses an embedding of the surface in $\mathbb{R}^3$ (same setting as on the slide before)
- Gauss' Theorema Egregium: Gaussian curvature depends only on the first fundamental form
- Alternative formulation: Gaussian curvature can be determined entirely by measuring angles, distances, and their rates directly on the surface without concern for any possible embidding in $\mathbb{R}^3$ $\rightarrow$ *Gaussian curvature is an intrinsic invariant of a surface*

Second Fundamental Form

Curvatures: Mean Curvature:

$$H = \frac{k_1 + k_2}{2}$$

- interesting for the analysis of minimal surfaces (zero mean curvature) and for the analysis of physical interfaces between fluids
- examples:
  - soap film: mean curvature zero
  - soap bubble: constant mean curvature
- in contrast to Gaussian curvature, H is an extrinsic property.
  Example:
  Plane and Cylinder are locally isometric but the plane has zero mean curvature whereas the cylinder does not

Second Fundamental Form

Second Fundamental Form:

Shape Operator: $L_u : T_uX \to T_uX : \quad L_u = -dN_u \circ X_u^{-1}$

$II_u(x, y) = \langle L_u(a), b \rangle, \quad a, b \in T_uX$

Matrices:

Shape Operator: $h_j^i = h_{jk} g^{ki}$

$II_u \; h_{ij} = -\langle X_i, N_j \rangle = +\langle X_{ij}, N \rangle$

Second Fundamental Form

Second Fundamental Form:

- basically, II is the normal curvature to a curve tangent to a surface X
- encodes extrinsic as well as intrinsic curvatures
- the shape operator has two real-valued eigenvalues: the principal curvatures
- the determinant of the matrix for II describes how the survace bends at a certain point similarly to the Gaussian curvature:
  - $h_{11}h_{22} - h_{12}^2 > 0$: elliptical (e.g. ellipsoid, sphere)
  - $h_{11}h_{22} - h_{12}^2 = 0$: parabolic (e.g. cylinder)
  - $h_{11}h_{22} - h_{12}^2 < 0$: hyperbolic (e.g. hyperboloid)

Second Fundamental Form

Exercise:
Compute II for a sphere.

Second Fundamental Form

Exercise:
Compute II for a sphere of radius $r > 1$.

Solution:

Sphere of radius $r > 0$: $X(u, v) = \begin{pmatrix} r\cos u \sin v \\ r \sin u \sin v \\ r \cos v \end{pmatrix}$. Field of unit normals: $\nu(u, v) = \frac{1}{r} X(u, v)$

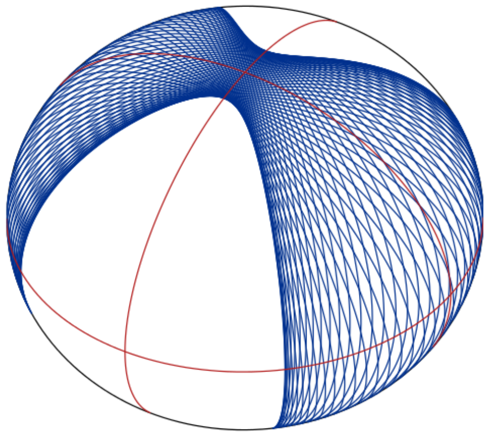
Second partial derivatives of X:

$X_{uu} = \begin{pmatrix} -r\cos u \sin v \\ r \sin u \sin v \\ 0 \end{pmatrix}$, $X_{uv} = X_{vu} = \begin{pmatrix} -r\sin u \cos v \\ r \cos u \cos v \\ 0 \end{pmatrix}$, and $X_{vv} = -X$

Coefficients: $h_{11} = -r \sin^2 v$, $h_{12} = h_{21} = 0$, and $h_{22} = -r$

Geodesics

- Generelization of the notion of a "straight line" to "curved spaces"
- defined as a curve whose tangent vectors remain parallel if transported along an affine connection
- in case of the Levi-Civita connection, geodesics are locally shortest paths between points in the space
- examples: lines in the plane, great circles on a sphere, triaxial ellipsoid: see image below



Geodesics

- shortest path between to points: write the equation for the length of a connecting curve and minimize it using variational calculus
- *problem:* possibly infinitely many ways to parameterize the shortest path
- *idea:* demand the curve to minimize the length and additionally be paremeterized with "constant velocity" $\rightarrow$ distance of points $C(t_1)$ and $C(t_2)$ on curve $C(t)$ is proportional to $|t_2 - t_1|$
- equivalently: *Energy*: elastic band stretched between two points contracts its length minimizing its energy $\rightarrow$ if this contraction is resptricted to the shape of the space (i.e. the band e.g. "stays in the plane"), the result is a geodesic
- note: Geodesics are similar to but not equal to shortest paths. **Exercise:** find a counterexample

Geodesics

Exercise:
Find a counterexample proving that geodesics are not the same as shortest distances.

Solution:
On a sphere, a geodesic is a great circle. Taking the "long way" from one point to another is also a geodesic but obviously not the shortest distance.