Algorithmic Geometry WS 2017/2018

Prof. Dr. Hans Hagen
Benjamin Karer M.Sc.

http://gfx.uni-kl.de/~alggeom

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Introduction

People

- Lecture: Prof. Dr. Hans Hagen,
  - 36-226
  - hagen@informatik.uni-kl.de
- Responsible for exams: Prof. Dr. Hans Hagen
- Exercises: Benjamin Karer M.Sc., karer@rhrk.uni-kl.de
  - 36-415
  - karer@rhrk.uni-kl.de

Lecture and Exercise

- Room: 36-265
- Wednesday, 10:00-11:30 and Friday, 11:45-13:15
- Demonstration of practical exercises in the lab (36-223)
- See homepage for news and changes

Exercise

- Exercise sheets will be uploaded on the homepage
- Deadlines:
  - Theoretical exercises: lecture on Wednesdays
  - Practical exercises: demonstration in the lab (36-223) until end of semester
- requirements for the exam: reasonable attempt to 100% of the exercises + summary and short talk for research paper
- registration: list here and via email to karer@rhrk.uni-kl.de

Paper Summary and Talk

- 1 paper (approx. 12 pages) each
- made available at about half of the lecture
- summary:
  - roughly 2 pages, *including* images
  - short, high-level summary of the paper's motivation, solution, and proclaimed results
  - focus on your own discussion of the paper:
    - is the motivation sufficient?
    - are the design decisions sound and well motivated?
    - are the conclusions justified by the results?
- paper talk:
  - at the end of the semester
  - 12 minutes, max. 15
  - even more high level description and discussion
  - after each talk, approx. 5 minutes of scientific discussion

Contents

- Interpolation
- Spline Curves
- Bézier Curves
- B-Spline Curves
- Gordon-Coons Patches
- Bézier and B-Spline Surfaces
- Curve and Surface Subdivision?

Literature

- G. Farin, **Curves and Surfaces for CAGD**, Academic Press, 1992.
- J. Hoschek, D.Lasser, **Fundamentals of CAGD**, A K Peters, Ltd. 1993.
- G. Farin, **NURBS for Curve and Surface Design from Projective Geometry**, 2nd edition, A K peters, Ltd. 1999.

Motivation

Motivation

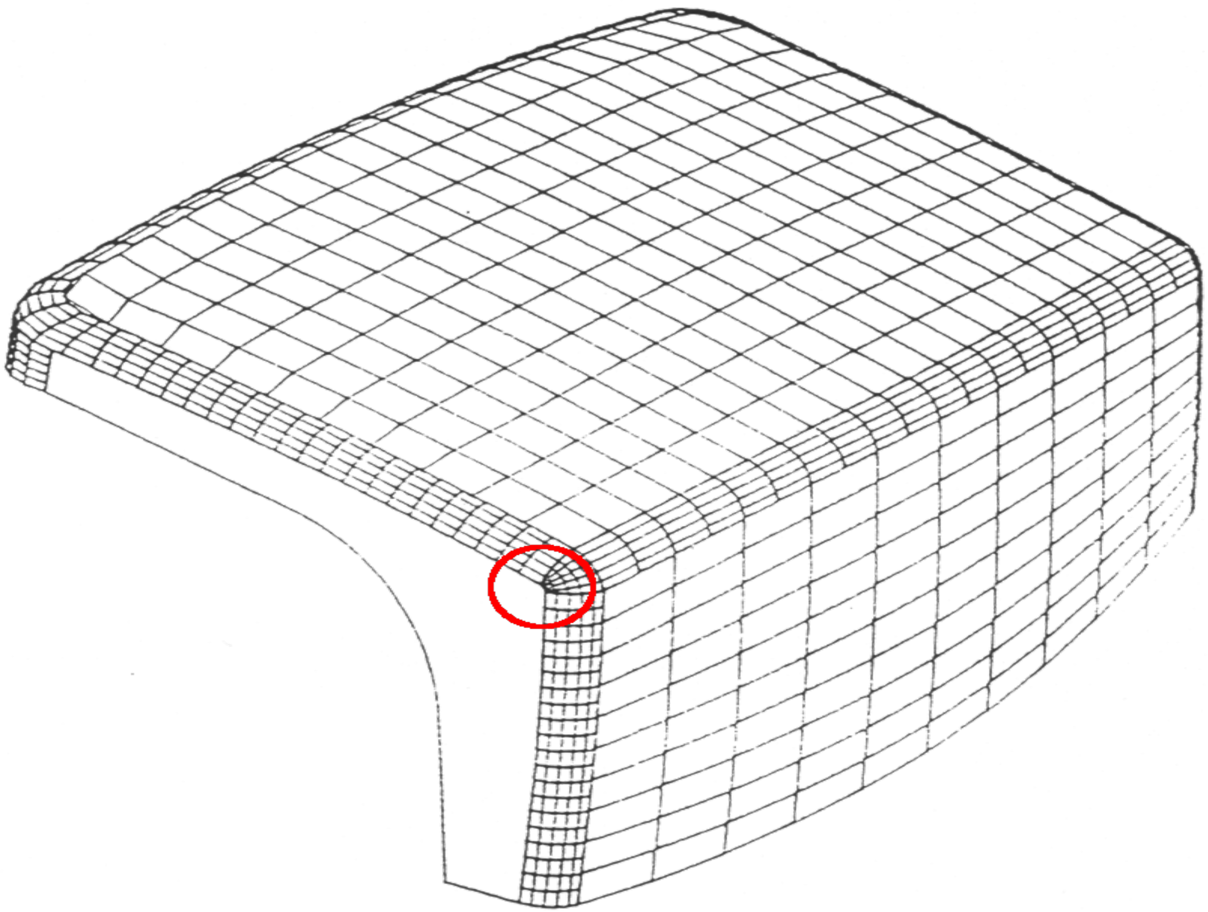

Figure: Segments of composite surfaces

Motivation

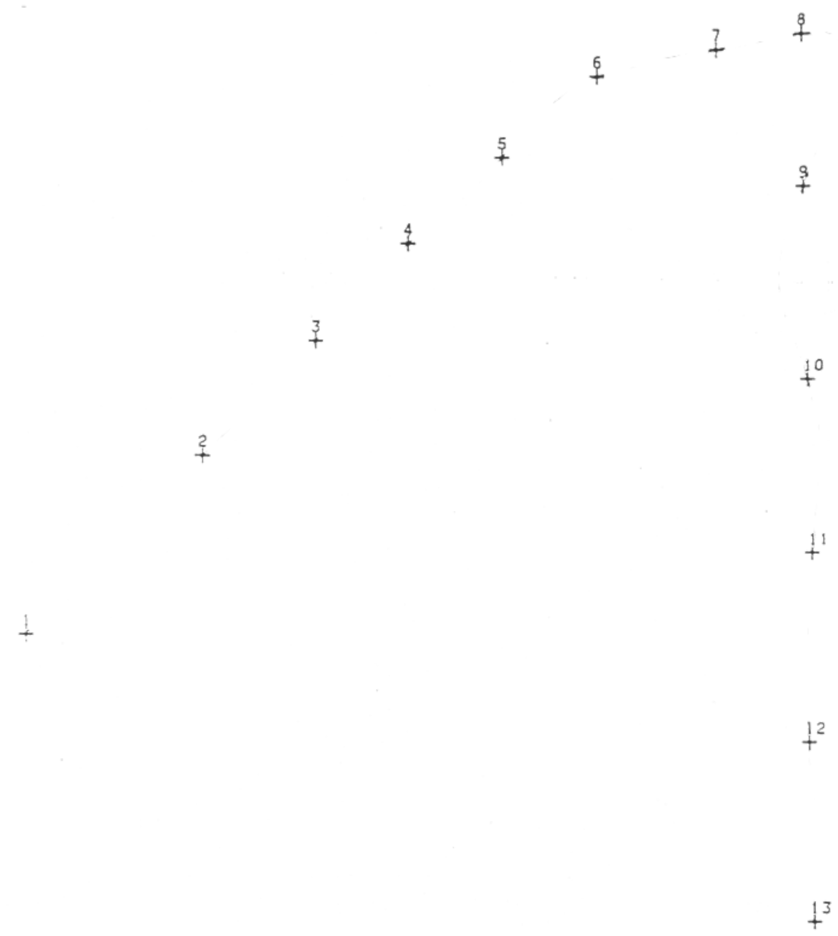

Figure: Interpolation

Motivation

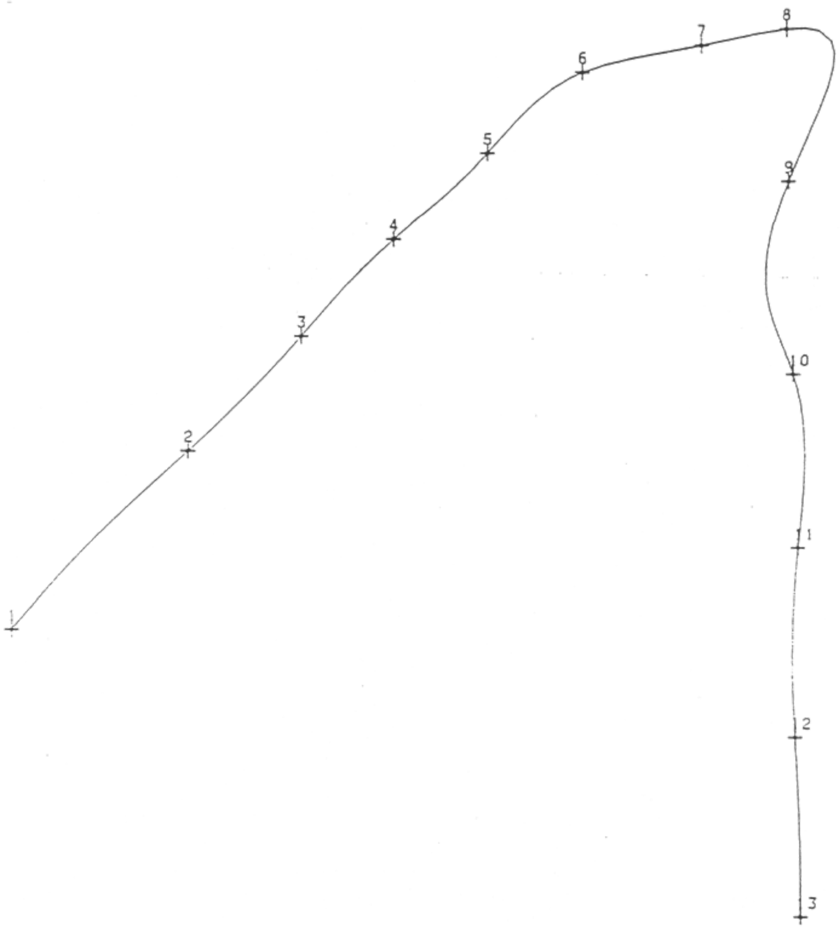

Figure: Interpolation (not shape preserving)

Motivation

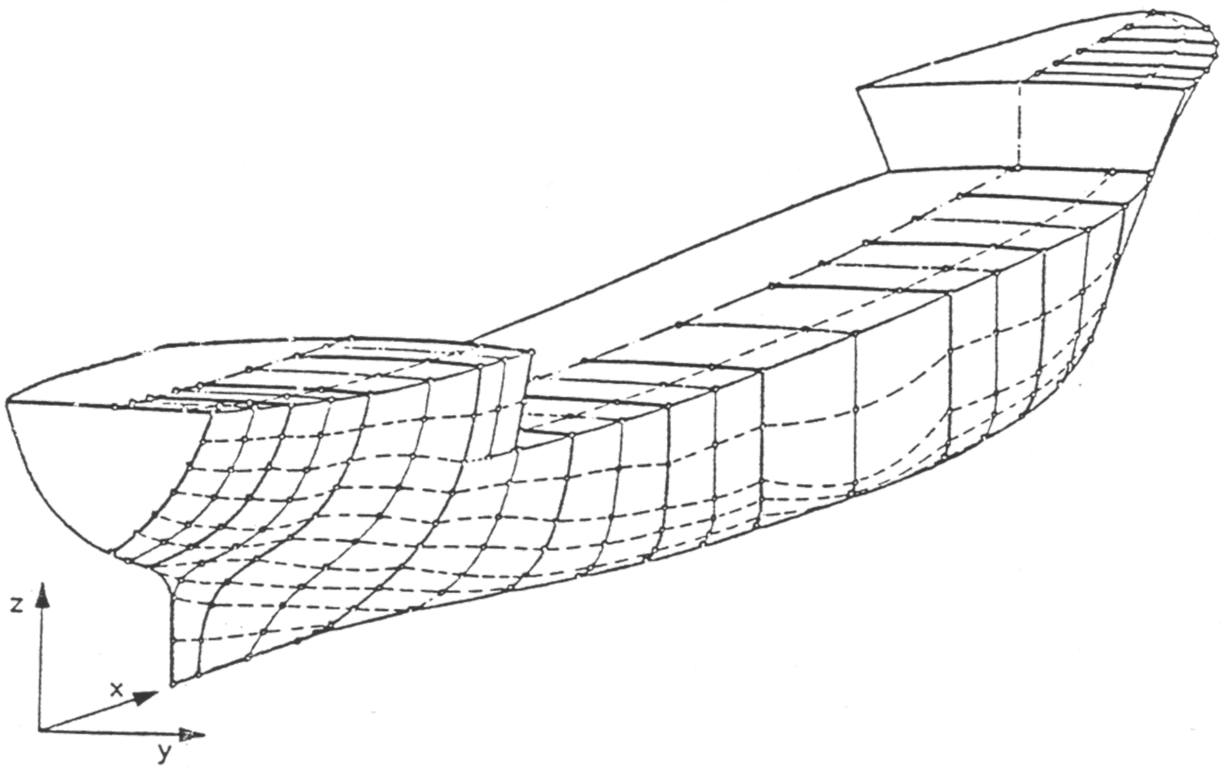

Figure: Piecewise smooth surface construction.

Motivation

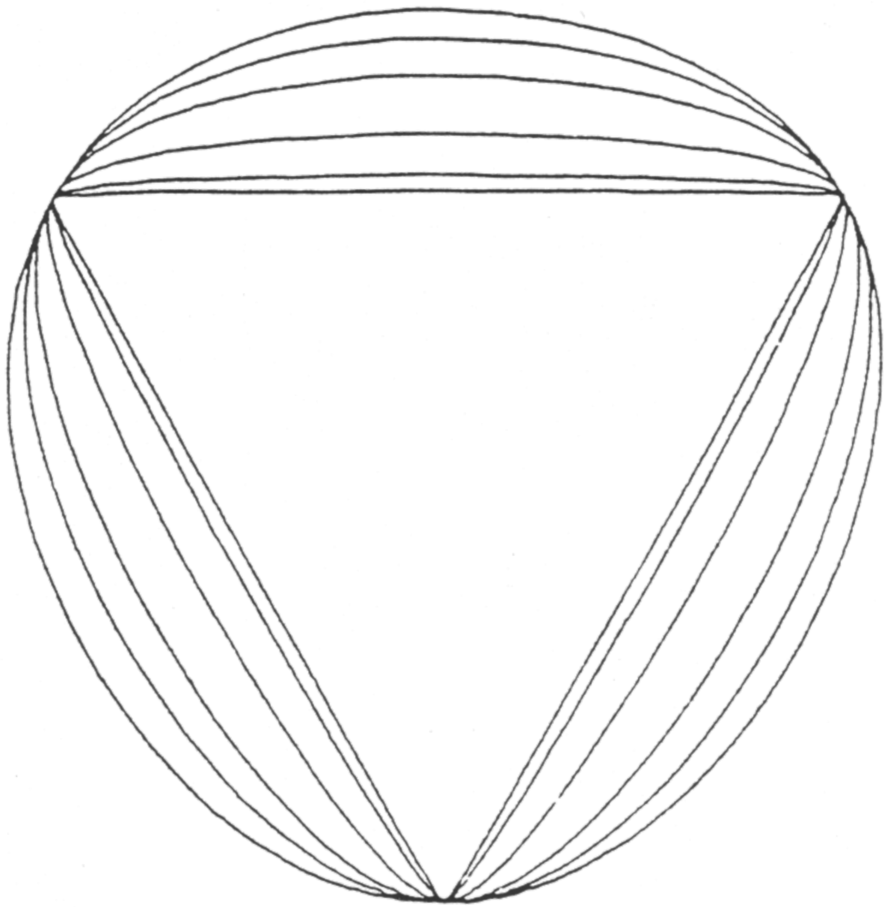

Figure: Multiple solutions to the same interpolation problem

Basics of Analytic Geometry

Vectors

- fundamental idea of analytic geometry: “calculate” geometric “facts”
- key technology: Vectors (with scalar and cross product)
- Vector: ordered pair of points: from P to Q:
  $\overrightarrow{a} = (a_1, a_2, a_3)^T = (q_1 - p_1, q_2 - p_2, q_3 - p_3)^T$
- Two vectors are equal if they are equal in direction and length
- Vectors build an algebraic group $(V, +)$ (they can be added)
- A vector space $(F, +, \cdot)$ over a field F is a set $(V, +)$ together with a scalar multiplication of elements from V (vectors) with elements from F (scalars)

Applications: Line

1. explicit vector form

$$\vec{r} = \vec{a} + t \cdot \vec{b}$$

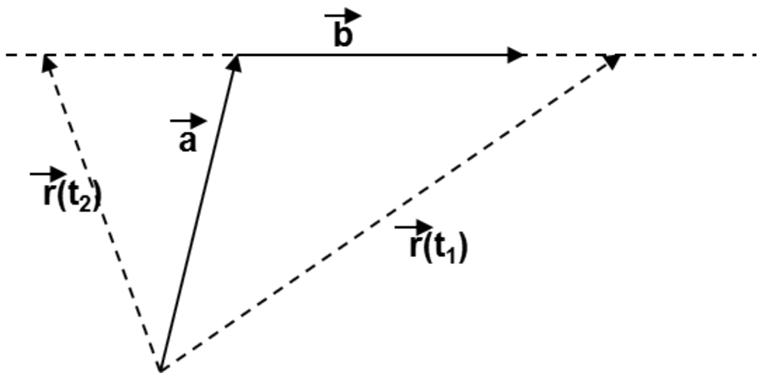

Figure: Line

2. parametric two point form

$$\vec{r} = \vec{a} + t \cdot \left(\vec{b} - \vec{a}\right)$$

Application: Plane

1. explicit point-vector form

$$\vec{r} = \vec{a} + t \cdot \vec{b} + \tau \cdot \vec{c}$$

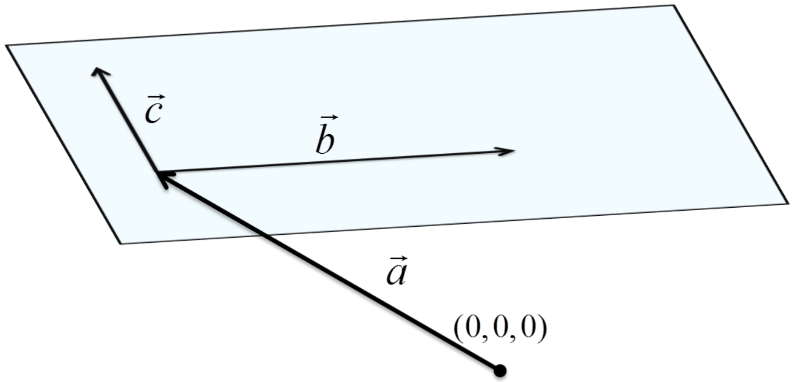

Figure: Plane

2. parametric three point form

$$\vec{r} = \vec{a} + t \cdot \left(\vec{b} - \vec{a}\right) + \tau \cdot (\vec{c} - \vec{a})$$

Linear Dependence

Definition

n vectors $\overrightarrow{a_1}, \ldots, \overrightarrow{a_n}$ are *linearly dependent*, if there are n scalars $\alpha_1, \ldots, \alpha_n$ which are not all zero, such that $\alpha_1 \overrightarrow{a_1} + \ldots + \alpha_n \overrightarrow{a_n} = 0$.

These vectors are called *linearly independent*, if there are no such scalars.

Fact

A pair of linearly dependent vectors is always parallel.
More than n vectors in a n-dimensional space are always linearly dependent.

Scalar Product

Definition

$$\langle , \rangle : V \times V \to \mathbb{R}$$

$$\left\langle \vec{a}, \vec{b} \right\rangle := a_1 b_1 + \ldots + a_n b_n$$

The scalar product of two vectors is the multiplication of the length of one vector times the length of the projection of the other vector onto this vector.

Scalar Product - Properties

1. $\|\overrightarrow{a}\| := \sqrt{\langle \overrightarrow{a}, \overrightarrow{a} \rangle}$ defines a norm $\| \, \| : V \longrightarrow \mathbb{R}_0^+$.
2. $\left\langle \overrightarrow{a}, \overrightarrow{b} \right\rangle = \|\overrightarrow{a}\| \cdot \left\|\overrightarrow{b}\right\| \cdot \cos\Phi$.
3. $\left\langle \overrightarrow{a}, \overrightarrow{b} \right\rangle = 0 \Leftrightarrow \overrightarrow{a} \perp \overrightarrow{b}$

Vector- / Cross-Product

Definition

$[,] : V \times V \to V; \;\; V \cong \mathbb{R}^3$

$$\left[\vec{a}, \vec{b}\right] := \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}; \; \{e_1, e_2, e_3\} \text{ basis of } \mathbb{R}^3$$

$[,] : V \times V \to V$ is a bilinear, anti-symmetric vector

Volume Product

Definition

$\left\langle \left[\vec{a}, \vec{b}\right], \vec{c} \right\rangle$ is the oriented volume spanned by $\vec{a}, \vec{b}, \vec{c}$.

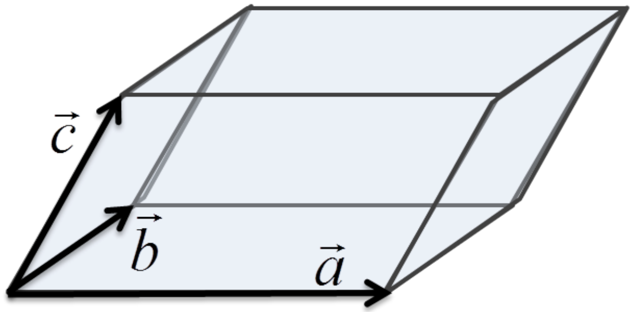

Figure: volume defined by vectors $\overrightarrow{a}$, $\overrightarrow{b}$, and $\overrightarrow{c}$

Properties

1. $\left[\vec{a}, \vec{b}\right] = 0 \Leftrightarrow \vec{a}, \vec{b}$ are linearly dependent.
2. $\left[\vec{a}, \vec{b}\right]$ is orthogonal to $\vec{a}$ and $\vec{b}$; $\left\{\vec{a}, \vec{b}, \left[\vec{a}, \vec{b}\right]\right\}$ is a right hand system.
3. $\left\|\left[\vec{a}, \vec{b}\right]\right\| = \|\vec{a}\| \cdot \left\|\vec{b}\right\| \cdot \sin\Phi = \sqrt{\left(\|\vec{a}\|^2 \cdot \left\|\vec{b}\right\|^2 - \left\langle \vec{a}, \vec{b}\right\rangle^2\right)}.$
4. $\left\langle \vec{c}, \left[\vec{a}, \vec{b}\right]\right\rangle = \det\left(\vec{c}, \vec{a}, \vec{b}\right) = \left|\vec{c}, \vec{a}, \vec{b}\right|.$
5. $\left\langle \left[\vec{a}, \vec{b}\right], \left[\vec{c}, \vec{d}\right]\right\rangle = \langle \vec{a}, \vec{c}\rangle \cdot \left\langle \vec{b}, \vec{d}\right\rangle - \left\langle \vec{a}, \vec{d}\right\rangle \cdot \left\langle \vec{b}, \vec{c}\right\rangle.$
6. $\left[\vec{a}, \left[\vec{b}, \vec{c}\right]\right] = \langle \vec{a}, \vec{c}\rangle \cdot \vec{b} - \left\langle \vec{a}, \vec{b}\right\rangle \cdot \vec{c}.$
7. $\left[\left[\vec{a}, \vec{b}\right], \left[\vec{c}, \vec{d}\right]\right] = \det\left(\vec{a}, \vec{b}, \vec{d}\right) \cdot \vec{c} - \det\left(\vec{a}, \vec{b}, \vec{c}\right) \cdot \vec{d}.$

Calculations with 3-dimensional Column-Vectors

Addition

$$\vec{a} + \vec{b} := \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}$$

Scalar Multiplication

$$\lambda \cdot \vec{a} = \begin{pmatrix} \lambda \cdot a_1 \\ \lambda \cdot a_2 \\ \lambda \cdot a_3 \end{pmatrix}$$

Scalar Product

$$\left\langle \vec{a}, \vec{b} \right\rangle = a_1 \cdot b_1 + a_2 \cdot b_2 + a_3 \cdot b_3$$

Calculations with 3-dimensional Column-Vectors

Vector Product

$$\left[\vec{a}, \vec{b}\right] = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

Volume Product

$$\left\langle \left[\vec{a}, \vec{b}\right], \vec{c} \right\rangle = \det \begin{pmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

Applications - Hesse form

Definition

P_1, P_2, P_3: Points on a plane

$$HF := \frac{[(P_2 - P_1), (P_3 - P_1)]}{\|[(P_2 - P_1), (P_3 - P_1)]\|} \rightarrow \langle (\vec{r} - P_1), HF \rangle = 0$$

Distances

- $\langle(\vec{a} - P_1), HF\rangle$ is the distance of a point to the plane.
- Distance of point P to straight line $r = \vec{a} + t \cdot \vec{b}$: $\frac{||[(\vec{p}-\vec{a}),\vec{b}]||}{||\vec{b}||}$

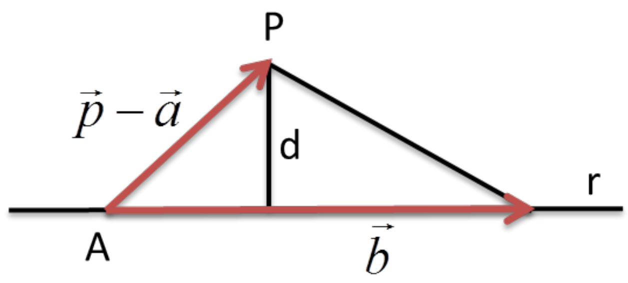

Figure: Distance point-to-line: $\frac{1}{2}d||\vec{b}|| = \frac{1}{2}||[\vec{p} - \vec{a}, \vec{b}]||$

- The non-intersecting straight lines $\vec{r} = \vec{a_1} + t \cdot \vec{b_1}$ and $\vec{s} = \vec{a_2} + \tau \cdot \vec{b_2}$ have the distance:
$d = \frac{\langle(\vec{a_1}-\vec{a_2}),[\vec{b_1},\vec{b_2}]\rangle}{||[\vec{b_1},\vec{b_2}]||}$, if $\det\left(\vec{a_1} - \vec{a_2}, \vec{b_1}, \vec{b_2}\right) \neq 0$

Distances

Locations of the points of shortest distance

$$\tau_0 = \frac{\det\big((\vec{a_1} - \vec{a_2}), \vec{b_1}, [\vec{b_1}, \vec{b_2}]\big)}{\big\langle [\vec{b_1}, \vec{b_2}], [\vec{b_1}, \vec{b_2}] \big\rangle}$$

$$t_0 = \frac{\det\big((\vec{a_2} - \vec{a_1}), \vec{b_2}, [\vec{b_1}, \vec{b_2}]\big)}{\big\langle [\vec{b_1}, \vec{b_2}], [\vec{b_1}, \vec{b_2}] \big\rangle}$$