Algorithmic Geometry WS 2017/2018

Prof. Dr. Hans Hagen
Benjamin Karer M.Sc.

http://gfx.uni-kl.de/~alggeom

Bezier and B-spline surfaces

Using the Bernstein operator $B_m(F;t) = \sum_{i=0}^{m} F(\frac{i}{m}) \cdot B_i^m(t)$ we can build the Boolean sum:

$$\begin{aligned} (B_m \oplus B_n)(F) &= (B_m + B_n - B_m \cdot B_n)(F) \\ &= \sum_{i=0}^{m} F(\frac{i}{m}, t) \cdot B_i^m(s) + \sum_{j=0}^{n} F(s, \frac{j}{n}) \cdot B_j^n(t) \\ &\quad - \sum_{i=0}^{m} \sum_{j=0}^{n} F(\frac{i}{m}, \frac{j}{n}) \cdot B_j^n(t) \cdot B_i^m(s) \end{aligned}$$

In the case of $F(t, \frac{i}{m}) = \sum_{j=0}^{m} F(\frac{i}{m}, \frac{j}{n}) \cdot B_j^n(t)$ and $F(s, \frac{j}{n}) = \sum_{i=0}^{m} F(\frac{i}{m}, \frac{j}{n}) \cdot B_i^m(s)$, we get:

$$(B_m \oplus B_n)(F) = \sum_{i=0}^{m} \sum_{j=0}^{n} F(\frac{i}{m}, \frac{j}{n}) \cdot B_j^n(w) \cdot B_i^m(u)$$

Definition

The so-called **Bézier surface segment** of order (m, n):
As control structure we do have a polyhedral surface:

$$\begin{aligned} P_{m,n}(u,w) \quad = \quad & b_{ij} \cdot (1 - (mu - i) + (nw - j) + (mu - i)(nw - j)) \\ & + b_{i,j+1} \cdot ((nw - j) - (mu - i)(nw - j)) \\ & + b_{i+1,j} \cdot ((mu - i) - (mu - i)(nw - j)) \\ & + b_{i+1,j+1} \cdot ((mu - i)(nw - j)), \end{aligned}$$

with $0 \leq i \leq m-1$, $0 \leq j \leq n-1$ and $u \in \left[\frac{i}{m}, \frac{i+1}{m}\right]$, $w \in \left[\frac{j}{n}, \frac{j+1}{n}\right]$.
The $\{b_{ij}\}$ are called the Bézier points and they constitute the so called **Bézier net**.

Bezier and B-spline surfaces

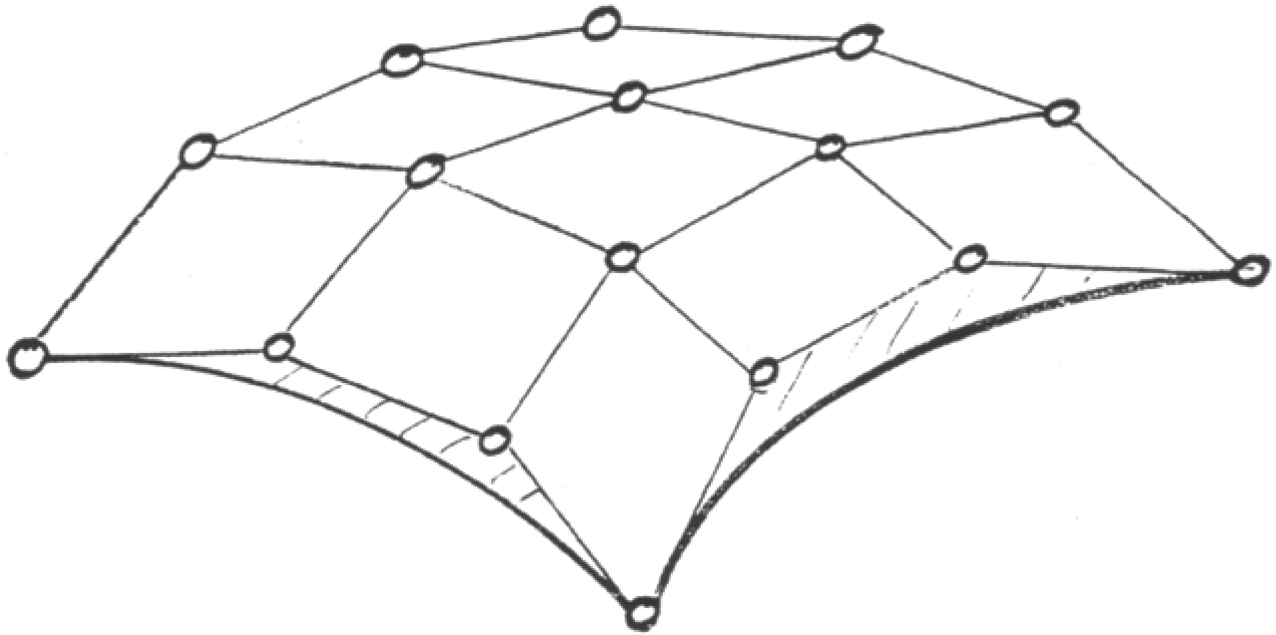

Figure: A cubic Bézier patch

We don't have twist problems.

Theorem

- The Bézier surface segment is contained by the convex hull of the Bézier net.
- The corner points of the net are the corner points of the surface.
- The boundary points of the net are the Bézier points of the boundary curve.
- A bezier surface segment is flat if the Bézier net is flat

Algorithm: Bézier surface segment

$$\sum_{i=0}^{m}\sum_{j=0}^{n} b_{ij} \cdot B_j^n(w_0) \cdot B_i^m(u_0) = \sum_{j=0}^{n} b_j^n \cdot B_j^n(w_0)$$

with $b_j^n := \sum_{i=0}^{m} \cdot B_i^m(u_0)$
To find the surface point at coordinates (u_0, w_0):

1. Use the algorithm for Bézier curves to determine b_j^n. These points are the Bézier points of surface lines in w-direction.
2. Again, use the algorithm for Bézier curves to find the point on the surface

Bezier and B-spline surfaces

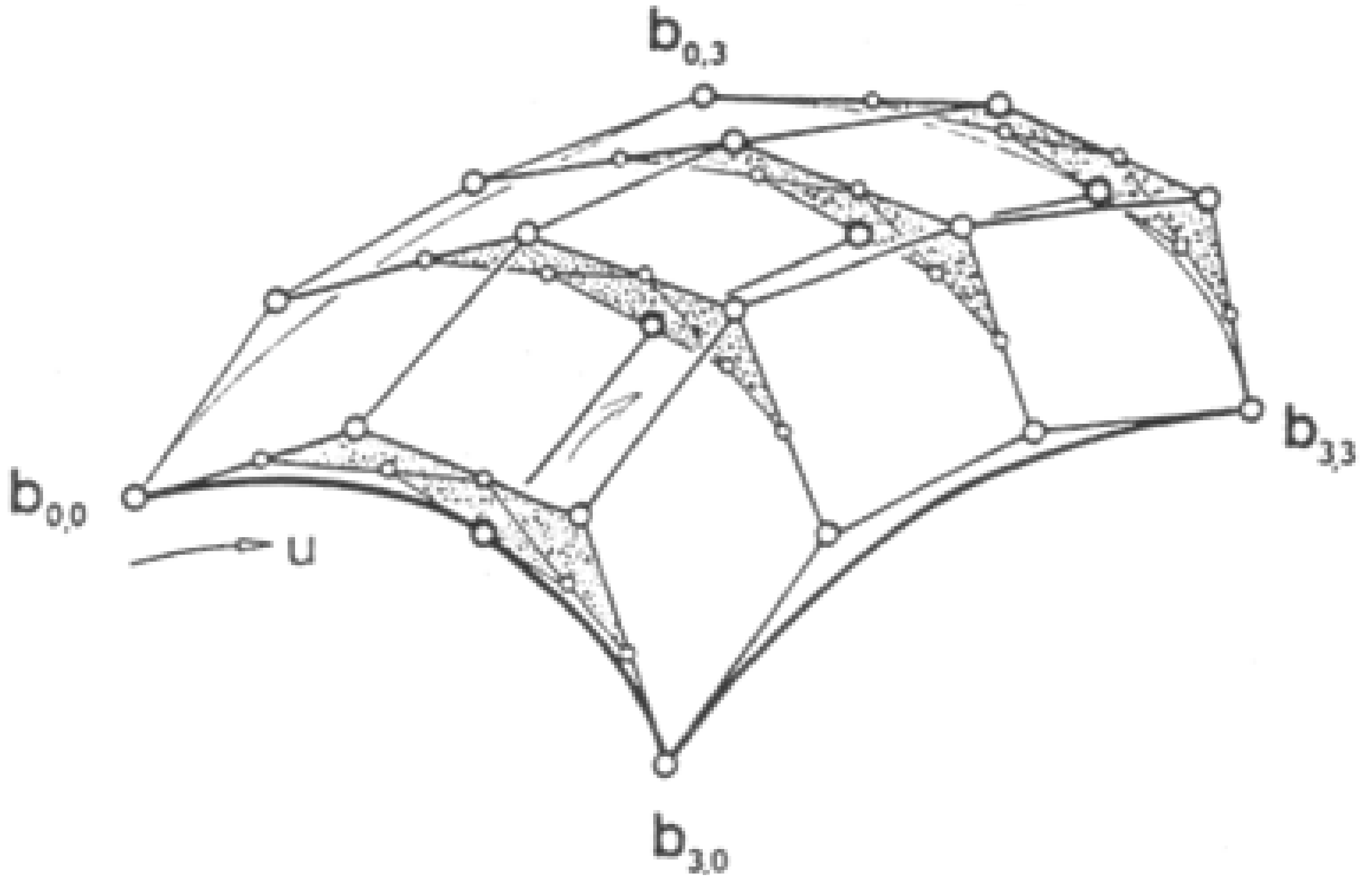

Figure: Illustration: finding points on a Bézier segment

Degree elevation of a Bézier surface segment

$$\widetilde{b_{ij}} = \frac{j}{n+1} \cdot \left(\frac{i}{m+1} b_{i-1,j-1} + \left(1 - \frac{i}{m+1}\right) b_{i,j-1} \right)$$
$$- \left(1 - \frac{j}{n+1}\right) \left(\frac{i}{m+1} b_{i-1,j} + \left(1 - \frac{i}{m_1}\right) b_{ij} \right),$$

with $0 < i < m+1$, $0 < j < n+1$ and $\widetilde{b_{ij}} = b_{ij}$ for $(ij) \in \{(0,0),(0,n+1),(m+1,n+1),(m+1,0)\}$.

Derivatives

$$\frac{\partial^p}{\partial u^p}\frac{\partial^q}{\partial w^q}X(u,w) = A_{pq}\cdot\sum_{i=0}^{m-p}\sum_{j=0}^{n-q}\Delta^{p,q}b_{ij}\cdot B_i^{m-p}(u)\cdot B_j^{n-q}(w)$$

$$= A_{pq}\cdot\sum_{i=0}^{m-p}\Delta^{p,0}\left(\sum_{j=0}^{n-q}\Delta^{0,q}b_{ij}\cdot B_j^{n-q}(w)\right)B_i^{m-p}(u),$$

with $A_{pq} := \frac{m!}{(m-p)!}\cdot\frac{1}{(b-a)^p}\cdot\frac{n!}{(n-q)!}\cdot\frac{1}{(d-c)^q}$, and
$\Delta^{p,q}b_{ij} = \Delta^{p,0}(\Delta^{0,q}b_{ij})$
There is the analogous convex hull property, but there is no variation diminishing property!

Convex Bézier surface segments

A Bézier surface segment of degree (n, m) is convex, if all Bézier points are vertices of the convex hull of the net and all straight lines $(b_{ij}, b_{i,j+1})$ to $(b_{ij}, b_{i+1,j})$ are edges of the convex hull. Each four Bézier points $b_{ij}, b_{i+1,j}, b_{i,j+1}, b_{i+1,j+1}$ form a parallelogram.

Bézier surface

A Bézier surface is a segmented surface. The surface segments $X_{p,q}$, $p = 0, \dots, k$, $q = 0, \dots, r$ of degree (m, n) over $[u_p, u_{p+1}] \times [w_q, w_{q+1}]$ are given by:

$$X_{p,q}(u, w) := \sum_{i=0}^{n} \sum_{j=0}^{m} b_{ij/pq} \cdot B_i^n \left(\frac{u - u_p}{u_{p+1} - u_p} \right) \cdot B_j^m \left(\frac{w - w_q}{w_{q+1} - w_q} \right), \tag{1}$$

The Bernstein polynomials $B_l^n(t) := \binom{n}{l}(1-t)^{n-l}t^l$, $0 \leq t \leq l$ are used as blending functions.

First order continuity

The conditions for C^1-continuity,
$\partial_u X_{pq}(u_{p+1}, w) = \partial_u X_{p+1,q}(u_{p+1}, w)$ and
$\partial_u X_{pq}(u_, wq+1) = \partial_u X_{p,q+1}(u_, wq+1)$, lead to the following condition on the Bézier points:

$$\frac{m}{\lambda_p} \cdot (b_{mj} - b_{m-1,j})_{pq} = \frac{m}{\lambda_{p+1}} \cdot (b_{ij} - b_{0j})_{p+1,q}\text{, and}$$

$$\frac{m}{\mu_p} \cdot (b_{im} - b_{i,m-1})_{pq} = \frac{m}{\mu_{p+1}} \cdot (b_{i1} - b_{i0})_{p,q+1}\text{, and,}$$

with $\lambda_p := u_{p+1} - u_p$ and $\mu_p := w_{q+1} - w_q$.

Bezier and B-spline surfaces

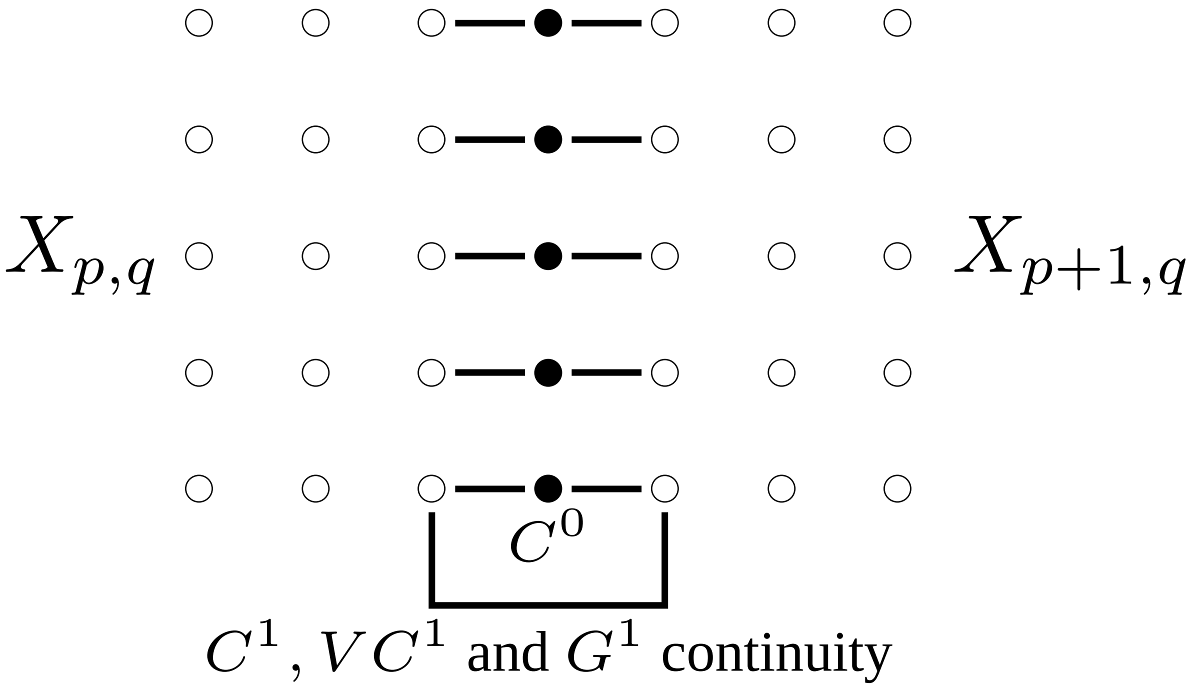

Figure: Two Bézier patches connected with C^0 continuity

A common boundary curve means: $b_{nj/pq} = b_{0j/p+1,q}$ and $b_{im/pq} = b_{i0/p,q+1}$.

Bezier and B-spline surfaces

As C^1 conditions we get:

$$
\begin{aligned}
b_{nj/pq} \cdot (\lambda_{p+1} + \lambda_p) &= \lambda_{p+1} b_{n-1,j/pq} + \lambda_p b_{1j/p+1,q} \qquad (2)\\
b_{im/pq} \cdot (\mu_{q+1} + \mu_q) &= \mu_{q+1} b_{i,m-1/pq} + \mu_q b_{i1/p,q+1}. \qquad (3)
\end{aligned}
$$

Using the parameterization as a design tool, we get **visual** C^1 **continuity**: "the first derivatives have the same direction".
With $\lambda := \frac{\lambda_p}{\lambda_{p+1}}$ and $\mu := \frac{\mu_q}{\mu_{q+1}}$:

$$
\begin{aligned}
b_{nj/pq} \cdot (1 + \lambda) &= b_{n-1,j/pq} + \lambda b_{1j/p+1,q}\\
b_{im/pq} \cdot (1 + \mu) &= b_{i,j-1/pq} + \mu b_{i1/p,q+1}
\end{aligned}
$$

Geometric G^1 continuity: "common tangent planes along a common boundary curve".

$$\det\left(\frac{\partial X_{pq}}{\partial u}, \frac{\partial X_{p+1,q}}{\partial u} \frac{\partial X_{p+1,q}}{\partial w}\right)\Bigg|_{(u_{p+1,w})} = 0,$$

This leads to:

$$\begin{aligned}
\frac{\partial X_{pq}}{\partial u}\Bigg|_{(u_{p+1},w)} &= \alpha(w)\,\frac{\partial X_{p+1,q}}{\partial u}\Bigg|_{(u_{p+1},w)} + \beta(u)\,\frac{\partial X_{p+1,q}}{\partial w}\Bigg|_{(u_{p+1},w)} \\
\frac{\partial X_{pq}}{\partial w}\Bigg|_{(u,w_{q+1})} &= \gamma(w)\,\frac{\partial X_{p,q+1}}{\partial w}\Bigg|_{(u,w_{q+1})} + \delta(w)\,\frac{\partial X_{p,q+1}}{\partial u}\Bigg|_{(u,w_{p+1})}
\end{aligned}$$

Remark: C^1 continuous Bézier surfaces have the same mixed partial derivatives along common boundary curves. This means the twist vectors are fixed by C^1 conditions!
This follows immediately out of

$$
\begin{aligned}
\left.\frac{\partial^2 X_{pq}}{\partial u \partial w}\right|_{(u_{p+1},w)} &= \frac{n}{\lambda_p}\cdot\frac{m}{\mu_q}\sum_{j=0}^{m-1}((b_{n,j+1}-b_{m-1,j+1})-(b_{mj}-b_{m-1,j}))_{pq}B_j^{m-1}(w) \\
\left.\frac{\partial^2 X_{p+1,q}}{\partial u \partial w}\right|_{u,w_{q+1}} &= \frac{n}{\lambda_p}\cdot\frac{m}{\mu_q}\sum_{j=0}^{m-1}((b_{1,j+1}-b_{0,j+1})-(b_{1j}-b_{0j}))_{p+1,q}B_j^m-1(w)
\end{aligned}
$$

and the equations (3).

Bezier and B-spline surfaces

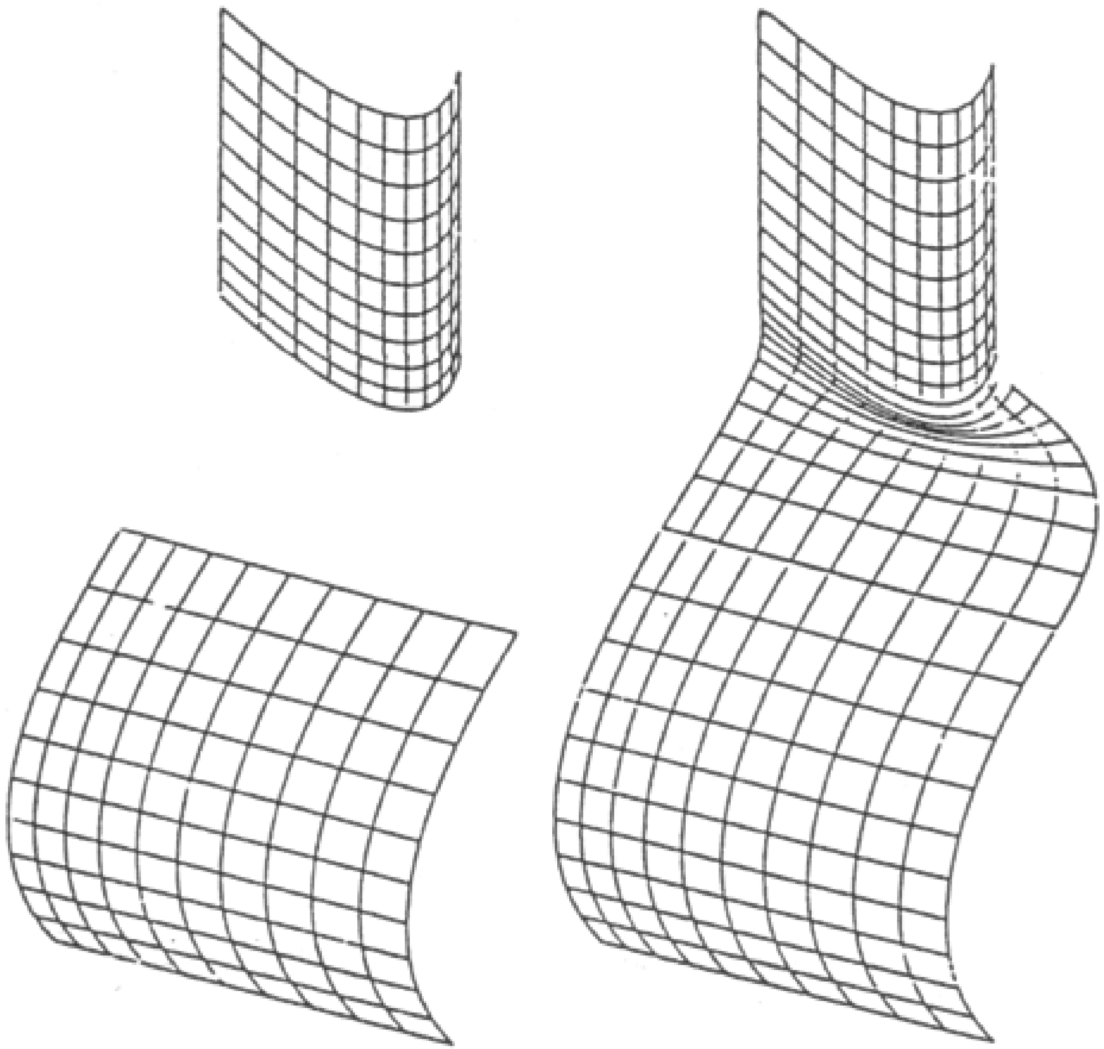

Figure: Application: connecting two separate surface patches

C^2 **continuity**:

$$\frac{\partial^2 X_{pq}}{\partial u^2}(u_{p+1}, w) = \frac{\partial^2 X_{p+1,q}}{\partial u^2}(u_{p+1,w}), \text{ and}$$
$$\frac{\partial^2 X_{pq}}{\partial w^2}(u, w_{q+1}) = \frac{\partial^2 X_{p,q+1}}{\partial w^2}(u, w_{q+1}).$$

Bézier point conditions:

$$\frac{n(n-1)}{\lambda_p^2}\cdot(b_{nj}-2b_{n-1,j}+b_{n-2,j})_{pq} = \frac{n(n-1)}{\lambda_{p+1}^2}\cdot(b_{2j}-2b_{ij}+b_{0j})_{p+1,q}$$
$$\frac{m(m-1)}{\lambda_q^2}\cdot(b_{i,m}-2b_{i,m-1}+b_{i,m-2})_{pq} = \frac{m(m-1)}{\lambda_{q+1}^2}\cdot(b_{i2}-2b_{i1}+b_{i0})_{p,q+1},$$

in addition to the first order conditions.

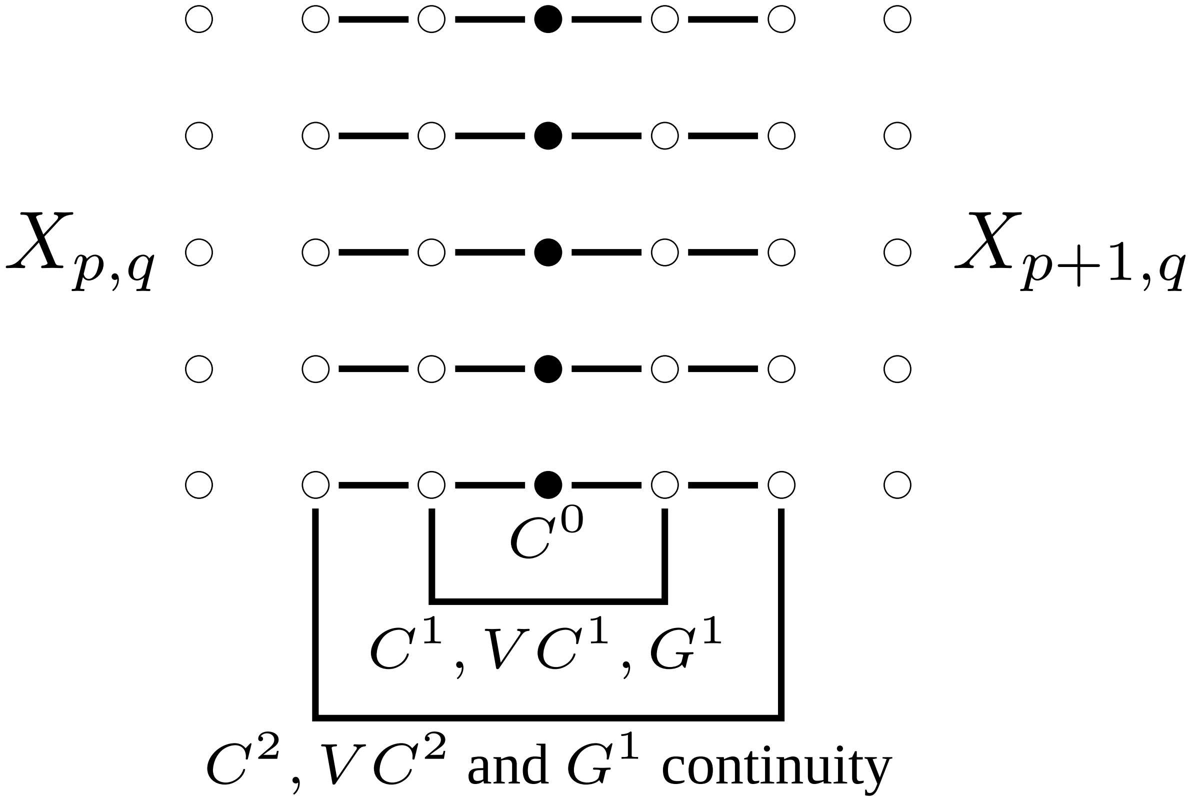

Figure: A visualization of the control points involved for different levels of continuity

Visual C^2 Continuity

For visual C^2 continuity, the second derivatives need to point in the same direction:

$$\begin{aligned}(b_{nj} - 2b_{n-1,j} + b_{n-2,j})_{pq} &= \lambda^2 (b_{2j} - 2b_{ij} + b_{0j})_{p+1,q}, \text{ and} \\ (b_{nj} - 2b_{n+1,j} + b_{n-2,j})_{qp} &= \mu^2 (b_{2j} - 2b_{i1} + b_{i0})_{q,p+1}\end{aligned}$$

with design parameters $\lambda := \frac{\lambda_p}{\lambda_{p+1}}$ and $\mu := \frac{\lambda_q}{\lambda_{q+1}}$.
Geometric C^2 conditions "create" design parameters to smooth surfaces.

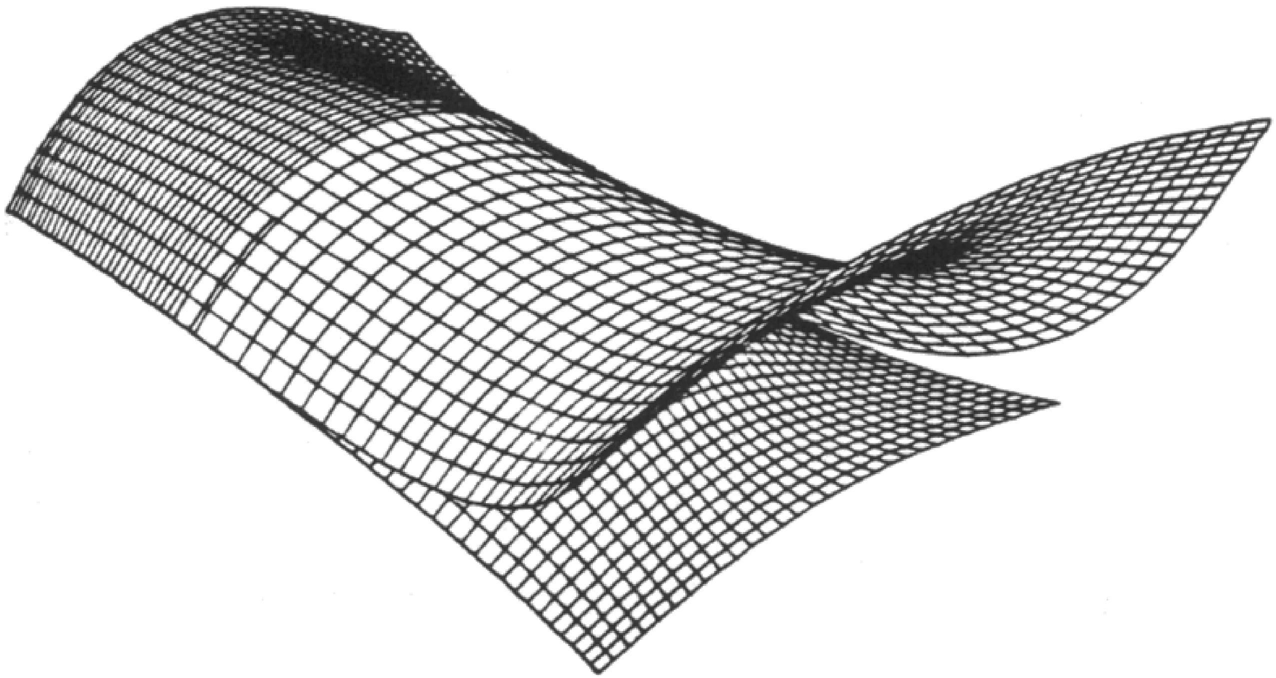

Figure: Two different continuations of a surface patch

Geometric G^2 continuity

For GC^2, normal section curvature values need to agree across the common boundary curves:

$$\begin{aligned}
\frac{\partial^2 X_{pq}}{\partial u^2}(u_{p+1}, w) &= \lambda^2 \frac{\partial^2 X_{p+1,q}}{\partial u^2}(u_{p+1}, w) + \nu_n \frac{\partial X_{p+1,q}}{\partial u}(u_{p+1}, w) \\
\frac{\partial^2 X_{pq}}{\partial w^2}(u, w_{q+1}) &= \mu^2 \frac{\partial^2 X_{q,p+1}}{\partial w^2}(u, w_{q+1}) + \nu_w \frac{\partial X_{p,q+1}}{\partial w}(u, w_{q+1})
\end{aligned}$$

General G^2 condition:

$$\frac{\partial^2 X_{pq}}{\partial u^2} = \alpha^2 \frac{\partial^2 X_{p+1,q}}{\partial u^2} + 2\alpha\beta \frac{\partial^2 X_{p+1,q}}{\partial u \partial w} + \beta \frac{\partial^2 X_{p+1,q}}{\partial w^2} + \gamma \frac{\partial X_{p+1,q}}{\partial u} + \delta \frac{\partial X_{p+1,q}}{\partial w}$$

Rational Bézier surfaces

Motivation: Quadrics are special rational surfaces

$$X(u,w) = \sum_{i=0}^{n}\sum_{j=0}^{m} b_{ij} \frac{\lambda_{ij}\cdot B_i^n(u)B_j^m(w)}{\sum_{i=0}^{n}\sum_{j=0}^{m}\lambda_{ij}B_i^n(w)}$$

$$= \sum_{i=0}^{n}\sum_{j=0}^{m} b_{ij}\frac{\lambda_{ij}}{R}B_i^n(u)B_j^m(w),$$

with

$$R = \sum_{\substack{i=0\\ i\neq\nu}}^{n}\sum_{\substack{j=0\\ j\neq\mu}}^{m}\lambda_{ij}B_i^n(u)B_j^m(w).$$

With $b_{\nu\mu}$ a point at infinity and $B_{\nu\mu}$ its direction, we get:

$$X(u,w) = \sum_{i=0}^{n}\sum_{j=0}^{m} b_{ij}\frac{\lambda_{ij}}{R_{\nu\mu}}B_i^n(u)B_j^m(w) + \frac{B_{\nu\mu}B_\nu^n(u)B_\mu^m(w)}{R_{\nu\mu}},$$

where

$$R_{\nu\mu} = \sum_{\substack{i=0\\ i\neq\nu}}^{n}\sum_{\substack{j=0\\ j\neq\mu}}^{m} \lambda_{ij}B_i^n(u)B_j^m(w)$$

Cylinder

$n = 2,\ m = 1,\ \lambda_{10} = \lambda_{11} = 0$, all other $\lambda_{ij} = 1$.

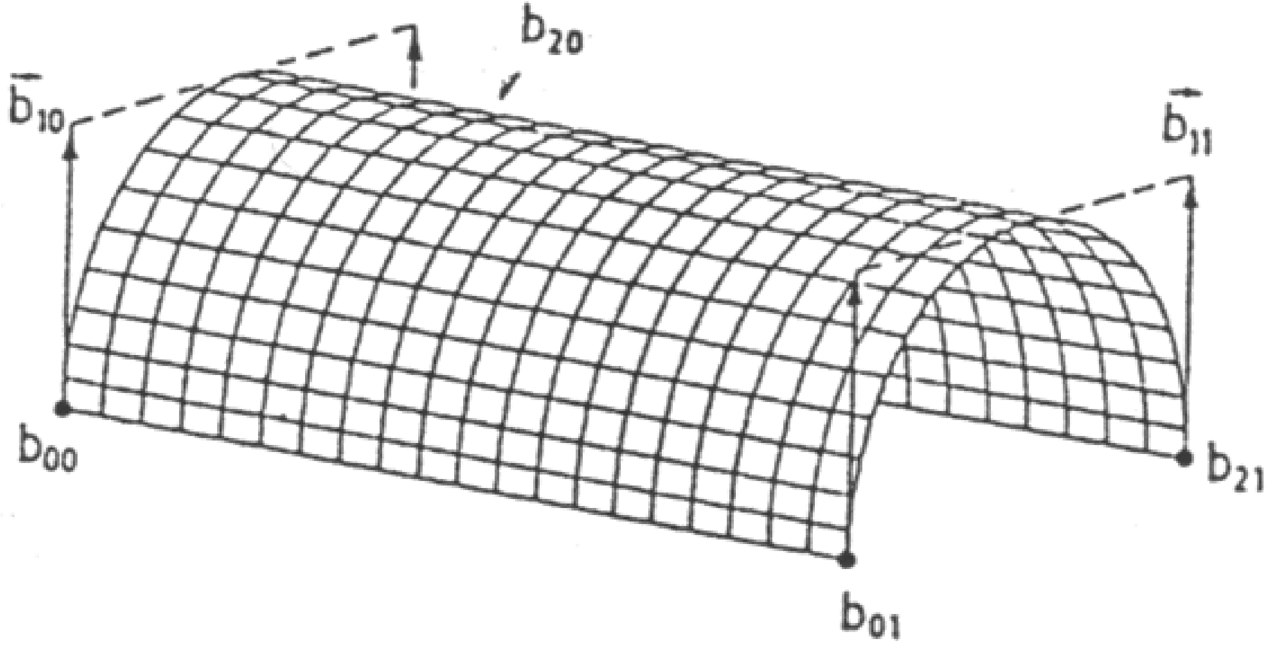

Figure: A half cylinder shell represented by a rational Bézier surface

$$
\begin{aligned}
X(u,w) \quad = \quad & \frac{b_{00} B_0^2(u) B_0^1(w) + b_{20} B_2^2(u) B_0^1(w)}{(B_0^2(u) + B_2^2(u))(B_0^1(w) + B_1^1(w))} \\
& + \frac{b_{01} B_0^2(u) B_1^1(w) + b_{21} B_2^2(u) B_1^1(w)}{(B_0^2(u) + B_2^2(u))(B_0^1(w) + B_1^1(w))} \\
& + \frac{b_{10} B_0^2(u) B_0^1(w) + b_{11} B_1^2(u) B_1^1(w)}{(B_0^2(u) + B_2^2(u))(B_0^1(w) + B_1^1(w))}
\end{aligned}
$$

In the special case of a circular cylinder:
$b_{00} = (r, 0, 0)$, $b_{01} = (r, 0, h)$, $b_{20} = (-r, 0, 0)$, $b_{21} = (r, 0, h)$,
$b_{10} = (0, r, 0)$, $b_{11} = (0, r, 0)$

Torus

$n = 2,\ m = 2,\ \lambda_{10} = \lambda_{21} = \lambda_{11} = \lambda_{01} = \lambda_{12} = 0$, all other $\lambda_{ij} = 1$.

$$
\begin{aligned}
X(u,w) \quad = \quad & \frac{1}{\tilde{R}}(b_{00} B_0^2(u) B_0^2(w) + b_{20} B_2^2(u) B_0^2(w) \\
& + b_{02} B_0^2(u) B_2^2(w) + b_{22} B_2^2(u) B_2^2(w) \\
& + b_{10} B_1^2(w) B_0^2(w) + b_{21} B_2^2(u) B_1^2(w) \\
& + b_{11} B_1^2(u) B_1^2(w) + b_{01} B_0^2(u) B_1^2(w) \\
& + b_{12} B_1^2(u) B_2^2(w))
\end{aligned}
$$

with $\tilde{R} = (B_0^2(u) + B_2^2(u))(B_0^2(w) + B_2^2(w))$

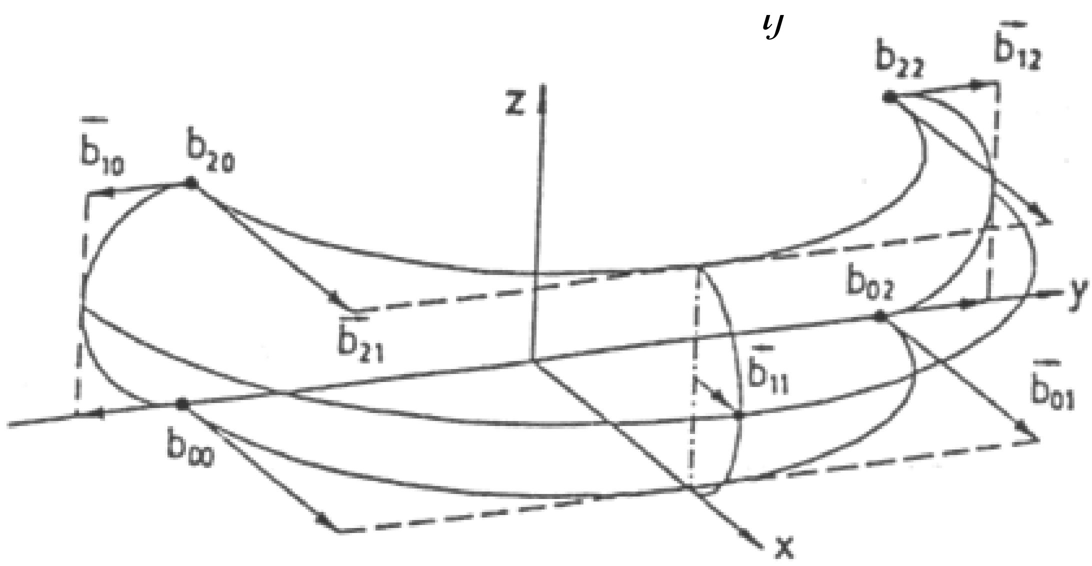

Figure: A half torus represented by a rational Bézier patch

For the special case of a (quarter-) circular torus:
$b_{00} = (0, -R, -r)$, $b_{20} = (0, -R, r)$, $b_{02} = (0, R, -r)$,
$b_{22} = (0, R, r)$, $b_{10} = (0, -r, 0)$, $b_{21} = (R, 0, 0)$, $b_{11} = (r, 0, 0)$,
$b_{01} = (R, 0, 0)$, $b_{12} = (0, r, 0)$

B-Spline surfaces

Control structure $d_{ij}(i = 0, \ldots, m \quad j = 0, \ldots, n)$:

$$X(u, w) = \sum_{i=0}^{m} \sum_{j=0}^{n} d_{ij} N_{i,j}(u) N_{j,l}(w)$$

To calculate a B-Spline surface we just run the algorithm for B-Spline curves twice:

$$\begin{aligned} X(u, w) &= \sum_{j=0}^{n} \left(\sum_{i=0}^{m} d_{ij} N_{i,k}(u) \right) N_{j,l}(w) \\ & \sum_{j=0}^{n} \widetilde{d}_j N_{j,l}(w) \quad \text{with } \widetilde{d}_j := \sum_{i=0}^{m} d_{ij} N_{i,k}(u) \end{aligned}$$

Algorithm for B-Spline surfaces

Input:

- de Boor net with vertices d_{ij} $(i = 0, \dots, m \quad j = 0, \dots, n)$
- knot vectors $\tau = \{t_0, \dots, t_{m+k}\}$ and $\mu = \{\widetilde{t}_0, \dots, \widetilde{t}_{m+l}\}$
- the order l/k of the spline

1. determine $\tilde{d}_j := \sum_{i=0}^{m} d_{ij} N_{i,k}(u)$ using the algorithm for spline curves
2. run the same algorithm with the now determined control points $\tilde{d}$: $X(u, w) = \sum_{j=0}^{n} \widetilde{d}_j N_{i,l}(w)$

The data exchange format for B-Spline surfaces is reduced to the transfer format for Bézier surfaces.